L19|L20 F09

Taylor polynomials, Taylor Series (Lagrange 1797) (59)

Taylor's theorem (7.14) Suppose $f', f'', \dots, f^{(n+1)}$ are defined on $[a,x]$ *. If we define

$$R_{n,a}(x) := f(x) - \left[f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right],$$

then $$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } \xi \in (a,x).$$

Pf. Fix x. For $t \in [a,x]$, define

$$S(t) = f(x) - \left[\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \right].$$

Note that S is diff'ble on (a,x), cts on $[a,x]$.

Taking $\frac{d}{dt}$ of both sides,

$$(f'(t)(x-t))' + \left[\frac{f''(t)}{2}(x-t^2)\right]' + \dots$$

$$S'(t) = -\Big[f'(t) + \left(f''(t)(x-t) - f'(t)\right) + \left(\frac{f'''(t)}{2}(x-t)^2 - 2\frac{f''(t)}{2}(x-t)\right) + \dots + \frac{f^{(n+1)}(t)}{n!}(x-t^n)\Big]$$

$$\therefore S'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Apply Cauchy MVT to S and $g(t) = (x-t)^{n+1}$

$$\therefore \exists \xi \in (a,x) \text{ s.t. } \frac{S(x)-S(a)}{g(x)-g(a)} = \frac{S'(\xi)}{g'(\xi)} = \dots$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

But $S(a) = R_{n,a}(x)$

$S(x) = R_{n,x}(x) = 0$!

Thus $f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{n!}(x-a)^{n+1}$ for some $\xi \in (a,x)$.

Def^n The polynomial

$$P_{n,a}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

is the Taylor polynomial of degree n for f at a.

e.g. $f(x) = e^x$, $a = 0$, $n = 2$ (Suppose we know $\exp' = \exp$.)

$p_2(x) = 1 + x + \frac{x^2}{2}$; hence $e^x = 1 + x + \frac{x^2}{2} + \frac{e^{\xi} \cdot x^3}{6}$; $\xi \in (0,x)$

e.g. $g(x) = \sin x$ ($\sin' = \cos$, $\cos' = -\sin$)

$n = 6$, $a = 0$ $g'(0) = 1$, $g''(0) = 0$, $g^{(3)}(0) = -1, \ldots$

$\therefore P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\cos(\xi)x^7}{7!}$, for some $\xi \in (0,x)$.

ie $\left| R_{6,0}(x) \right| = \left| -\frac{\cos(\xi)x^7}{7!} \right| \leq \frac{|x|^7}{7!}$. Thus, for $x = \frac{\pi}{10}$,

$\left| R_6\left(\frac{\pi}{10}\right) \right| \leq \frac{1}{7!}\frac{\pi^7}{10^7} < \frac{1}{10^7}$. Hence the Taylor poly. $P_6(x)$ (which is, in this case, of degree 5!) gives $\sin x$ correct to 7 decimal places, $\forall x$ s.t. $|x| \leq \frac{\pi}{10}$.

Taylor Series:

Suppose $U \subseteq \mathbb{R}$ is open, $[a,x] \subset U$, $f : U \to \mathbb{R}$ and $f^{(n)}$ exists $\forall n$ on U.

Def^n The Taylor series for f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Note that by Taylors theorem,

$$f(x) = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{N^{th}\text{ partial sum of the Taylor series for } f \text{ at } c} + R_{N,a}(x) \quad , \forall N \in \mathbb{N}.$$

<u>Def^n</u> We say the Taylor series converges to $f(x)$ if

$$\lim_{N\to\infty} R_N(x) = 0.$$

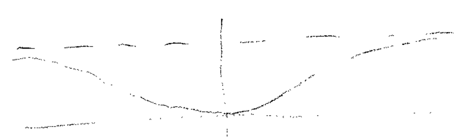
<u>Remark</u>: The Taylor series may converge at a point x, but not to $f(x)$! (This simply means $\lim_{N\to\infty} R_{N,a}(x)$ exists but is <u>not</u> zero. We'll see an example in a moment.)

e.g. $f(x) = \frac{1}{1-x}$; $a=0$; $f^{(k)}(0) = k!$ (ex.)

Hence, the Taylor series for f at 0 is $\sum_{n=0}^{\infty} x^n$, and we know $\sum_{n=0}^{\infty} x^n$ converges if $|x|<1$, and indeed it converges to $\frac{1}{1-x} = f(x)$! Hence, the Taylor series for f at 0 converges to f $\forall x \in (-1,1)$.

Note: f is defined on $[2,3]$, but the Taylor series for f at 0 will not converge (at <u>all</u>) for $x \in [2,3]$.

e.g. Define $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$



Then, $f'(x) = \frac{2}{x^3}, \dots, f^{(n)}(x) = \underset{\text{poly}}{p_n}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$ $(x \neq 0)$, so $f^{(n+1)}(0) = \lim_{h\to 0} \frac{p_n\left(\frac{1}{h}\right) e^{-h^2}}{h}$

$= \lim_{h\to 0} q\left(\frac{1}{h}\right) e^{-h^2} \underset{ex.}{=} 0$. Hence $f^{(n)}(0) = 0$, $\forall n$. Thus $\sum_{n=0}^{\infty} \frac{f^{(n)}(0) x^n}{n!} = 0 \underset{\circ}{!} = f(x)$.

So the Taylor series of f at 0 converges, but not to f.

e.g. $f(x) = \sum_{n\geq 0} \frac{\cos(2^n x)}{n!}$ is C^∞ on $\mathbb{R}$ but the Taylor series for f at 0 does not even converge!

$$\left(= \sum_{n=0}^{\infty} \frac{e^{2^{2n}} - 1}{(2n)!} x^{2n} . \right)$$

How do we know f is C^∞?

<u>Theorem</u> Suppose $\sum_{n\geq 0} a_n x^n$ converges at $x_0 \in \mathbb{R}$. Then,

$\forall \rho$, $0 \leq \rho < |x_0|$,

① $\sum a_n x^n$ converges uniformly on $[-\rho, \rho]$,

② $\sum n a_n x^{n-1}$ converges uniformly on $[-\rho, \rho]$,

& ③ $\left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}$!

Pf. Since $\sum a_n x_0^n$ converges, the terms are bdd. Hence $\exists M$ s.t. $|a_n||x_0|^n \leq M$, $\forall n \in \mathbb{N}$. Thus,

$$|x| \leq \rho \Rightarrow |a_n x^n| \leq |a_n| \rho^n = |a_n| |x_0|^n \frac{\rho^n}{|x_0|^n} \leq M\left(\frac{\rho}{|x_0|}\right)^n, \forall n \in \mathbb{N}$$

But $\left|\frac{\rho}{x_0}\right| < 1$, so $\sum_{n\geq} M\left|\frac{\rho}{x_0}\right|^n$ converges. By the Weierstrass M-Test, $\sum a_n x^n$ converges uniformly on $[-\rho, \rho]$. So ① holds

Moreover, $|x| \leq \rho \Rightarrow |n a_n x^{n-1}| \leq n|a_n| \rho^{n-1} = \frac{n|a_n||x_0|^n}{\rho} \frac{\rho^n}{|x_0|^n} \leq \frac{nM}{\rho} \left|\frac{\rho}{x_0}\right|^n$.

Let $r := \left|\frac{\rho}{x_0}\right|$. The limit ratio test on $\sum n r^n$ $\left(\frac{(n+1)}{n} \frac{r^{n+1}}{r^n} = (1+\frac{1}{n}) r \to r < 1 \right)$ shows that it converges. Hence so does $\sum \frac{M}{\rho} n r^n$, and by W-M-Test, $\sum n a_n x^n$ converges uniformly on $[-\rho, \rho]$.

Finally, since we have uniform convergence on $[0,x]$, $|x| \le \rho$,

$$\lim_{N\to\infty} \int_0^x \left(\sum_{n=0}^{N} n a_n t^{n-1}\right) dt = \int_0^x \lim_{N\to\infty} \sum_{n=0}^{N} n a_n t^{n-1}\, dt$$

$$= \int_0^x \left(\sum_{n=0}^{\infty} n a_n t^{n-1}\right) dt \qquad **$$

But $\displaystyle\int_0^x \sum_{n=0}^{N} n a_n t^{n-1}\, dt = \sum_{n=0}^{N} a_n x^n - a_0$

Hence $** \Rightarrow \displaystyle\sum_{n=0}^{\infty} a_n x^n - a_0 = \int_0^x \left(\sum_{n=0}^{\infty} n a_n t^{n-1}\right) dt$

By the FTC ($g(t) := \displaystyle\sum_{n=0}^{\infty} n a_n t^{n-1}$, as a uniform limit on $[0,x]$ of cts fns is cts on $[0,x]$), $\left(\displaystyle\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Cor 1 Suppose $f(x) = \displaystyle\sum_{n\ge 0} a_n x^n$, $|x| < R$.

Then $a_n = \dfrac{f^{(n)}(0)}{n!}$, $\forall n \ge 0$

Cor 2 $(e^x)' = e^x$, $\forall x \in \mathbb{R}$

exercise (Hadamard 1892) If $\rho = \lim_{n\to\infty} \left(\sup\{\sqrt[k]{|a|_k} \mid k \ge n\}\right)$, and $R = \frac{1}{\rho}$,

1. $\displaystyle\sum_{n\ge 0} a_n x^n$ & $\displaystyle\sum_{n\ge 0} n a_n x^{n-1}$ conv. $\forall x$, $|x| < R$
2. $\left(\displaystyle\sum_{n\ge 0} a_n x^n\right)' = \sum_{n\ge 0} n a_n x^{n-1}$