III 6 Differentiable function

<u>Def^n I</u> (Cauchy) 1821 Suppose $f : [a,b] \to \mathbb{R}$, $x_0 \in (a,b)$

We say <u>f is differentiable at x_0 if</u>

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{exists.} \qquad *$$

When it does, we denote the limit by $f'(x_0)$ and say "$f'(x_0)$ exists" or f' exists at x_0. ~~then~~ If f is differentiable $\forall x_0 \in (a,b)$ we say f is diff'ble on (a,b).

<u>Rmk</u> If $\lim\limits_{\substack{h \to 0 \\ h \geq 0}} \frac{f(a+h) - f(a)}{h}$ exists, we denote it by $f'_+(a)$

$\lim\limits_{h \to 0+}$

Similarly, we can define $f'_-(b) = \lim\limits_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$ when it exists;

if f is diff'ble on $[a,b)$ and both $f'_+(a)$, $f'_-(b)$ exist, we say f is diff'ble on $[a,b]$.

Fix $x_0 \in (a,b)$

<u>N.B.</u> If we define $r : [a,b] \to \mathbb{R}$ by

$$r(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0), & x \neq x_0 \\ 0, & x = x_0 \end{cases}$$

Then $*$ is equivalent to $\lim\limits_{x \to x_0} r(x) = r(x_0)$, i.e. r is cts <u>at x_0</u>.

<u>Proposition</u> (6.2) P233

Then (W 1861) f is diff'ble at x_0 iff

1) $\exists$ real number $f'(x_0)$

2) $\exists$ A function r (defined locally around x_0) with $r(x_0) = 0$, and <u>continuous at x_0</u>

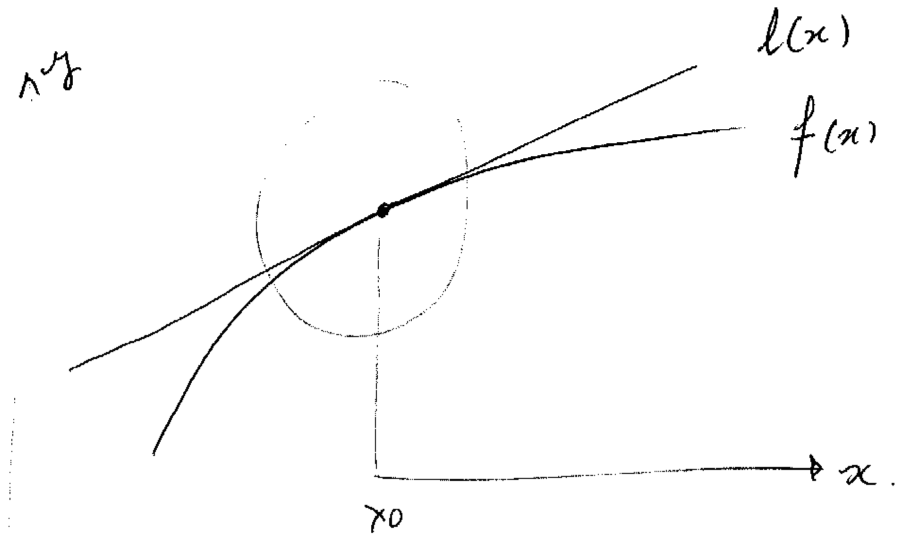
s.t.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)(x - x_0).$$

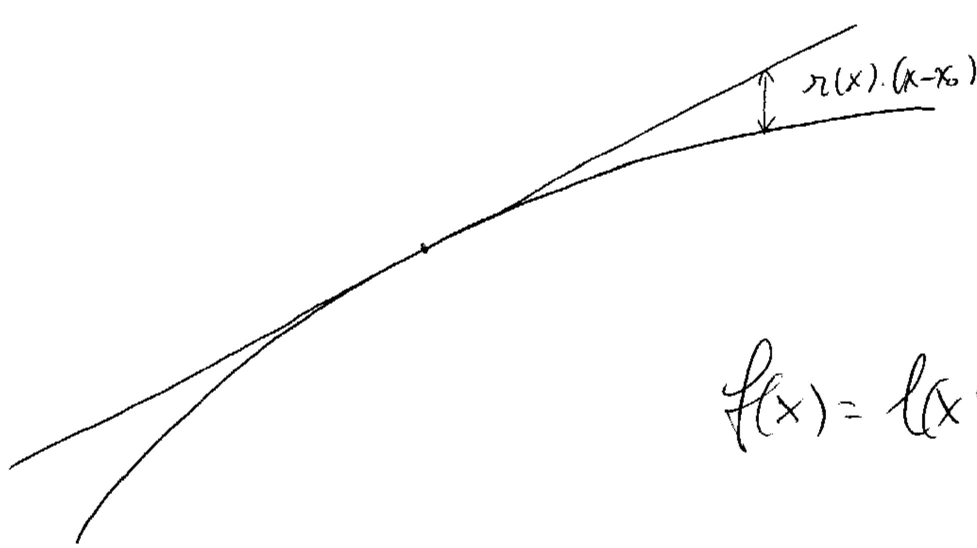
$(h = x - x_0)$ $\qquad f(x_0+h) = f(x_0) + f'(x_0)h + r(x_0+h)h$

<u>Remark</u>: Let $l(x) = f(x_0) + f'(x_0)(x - x_0)$

Then $y = l(x)$ is the tangent line to the graph of $y = f(x)$ at x_0!



$l : \mathbb{R} \to \mathbb{R}$ is "affine". (nearly linear)



$$f(x) = l(x) + \underbrace{r(x)(x - x_0)}_{\text{"remainder" term.}}$$

N.B. $f(x_0+h) = f(x_0) + \underbrace{f'(x_0)h}_{\text{linear in } h} + r(x_0+h)h$

linear in h, i.e. $h \mapsto f'(x)h$ is a linear function $\mathbb{R}\to\mathbb{R}$

The derivative at x_0 gives the best linear (affine) approximation to f near x_0

So f is diff'ble at x_0 iff there is a linear transformation $L:\mathbb{R}\to\mathbb{R}$ s.t.

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-L(h)}{h} = 0$$

$\Leftrightarrow$ $\exists$ linear map $L:\mathbb{R}\to\mathbb{R}$ and a f^n r, cts at x_0 with $r(x_0)=0$, s.t.

$$f(x_0+h) = f(x_0) + L(h) + r(x_0+h)h$$

$(\Rightarrow)$
$$f(x)-f(x_0) = L(x-x_0) + r(x)(x-x_0).$$

<u>Def^n</u> (Differential functions $\mathbb{R}^n\to\mathbb{R}^m$) Suppose U is open in $\mathbb{R}^n$, $f:U\to\mathbb{R}^m$.

Then, f is differentiable at $v_0\in U$ if there is a linear mapping $L:\mathbb{R}^n\to\mathbb{R}^m$ s.t.

$$\lim_{h\to 0} \frac{f(v_0+h)-f(v_0)-L(h)}{\|h\|} = 0 \quad \Big| \quad \lim_{v\to v_0} \frac{f(v)-f(v_0)-L(v-v_0)}{\|v-v_0\|} = 0.$$

<u>Propn</u> (3.3) Stolz 1887, Fréchet 1900) $f:U\to\mathbb{R}^m$ is differentiable at $v_0\in U$ iff

(1) $\exists$ a linear map $L:\mathbb{R}^n\to\mathbb{R}^m$

(2) $\exists$ a function $r: U\to\mathbb{R}^m$, cts at v_0 with $r(v_0)=0$, s.t.

$$f(v_0+h) = f(v_0) + L(h) + r(v_0+h)\|h\|$$

or
$$f(v) = f(v_0) + L(v-v_0) + r(v)\|v-v_0\|.$$

The linear map L is called the derivative of f at x_0. $L = df(x_0) = Df(x_0)$

<u>Thm</u> If $f'(x_0)$ exists, then f is cts at x_0.

Pf. The function $v \mapsto f(v_0) + L(v - v_0) + r(v)\|v - v_0\|$ will be cts at v_0 if $L : \mathbb{R}^n \to \mathbb{R}^m$ is cts (and hence uniformly cts on $\mathbb{R}^n$ as we know from ass # 4). So let $L: \mathbb{R}^n \to \mathbb{R}^m$ be any linear map. Then we know $\exists\, A \in M_{mn}(\mathbb{R})$ s.t. $L(v) = Av$. If $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, $A = [a_{ij}]$, $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n, \end{matrix}$ then $L(v) = \begin{bmatrix} b_1(v) \\ \vdots \\ b_m(v) \end{bmatrix}$ where

$b_i(v) = \sum_{j=1}^{n} a_{ij} x_j$. L is cts $\overset{⊛}{\Leftrightarrow}$ each $b_i : \mathbb{R}^n \to \mathbb{R}$ is But $p_i : \mathbb{R}^n \to \mathbb{R}$ $p_i\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = x_i$ is ($p_i \in (\mathbb{R}^n)^*$!)

continuous, since $\| p_i(v) - p_i(a) \| = |x_i - a_i| \le \|v - a\|$. ($a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$) ("$\delta = \varepsilon$") Any constant f^n is cts, and the product of cts f^ns is cts, and the sum of products of cts f^ns are cts, so L is cts

Hence f is cts at x_0 if f is diff'ble at x_0. ∎

---

e.g. $g(x) = \begin{cases} 0, & x = 0, 1, -1, \text{ or } x \in [-1,1] \setminus \mathbb{Q} \\ \frac{1}{q^2}, & x = \frac{p}{q}, \ \gcd(p,q) = 1. \end{cases}$

Note that $|g(x)| \le |x|^2$, so $\lim_{h \to 0} \frac{g(h)}{h} = 0$. Hence g is diff'ble at 0 but is discontinuous at <u>every</u> pt in $[-1,1] \cap \mathbb{Q}$ except 0 !

⊛ This is true because $\|L(v)\| = \sqrt{b_1^2 + \cdots + b_m^2} \le m \left(\max_{1 \le j = 1, m} |b_j| \right)$

and $|b_i(v)| \le \|L(v)\|$, $\forall\, i,\ 1 \le i \le m$

---

e.g. $f(x,y) = x^2 + y^2$; $h = (h_1, h_2)$

$f(x_0 + h_1, y_0 + h_2) = x_0^2 + h_1^2 + 2x_0 h_1 + y_0^2 + 2y_0 h_2 + h_2^2$

$= f(x_0, y_0) + 2(x_0, y_0) \cdot (h_1, h_2) + \|h\|^2$

Here, $L(h) = 2(x_0, y_0) \cdot h$ $\quad = f(x_0, y_0) + 2(x_0, y_0) \cdot h + \|h\| \cdot \|h\|$

$r(v_0 + h) = \|h\|$. ; $f(v) - f(v_0) = 2v_0 \cdot (v - v_0) + \|v - v_0\| \|v - v_0\|$

<u>Theorems</u> (6.6, 6.7, 6.8) Assuming f, g are defined near x_0 and the following expressions make sense!

(I)

If $f'(x_0)$, $g'(x_0)$ exist, so does $(f+g)'$, $(fg)'$ $\left(\frac{f}{g}\right)'$ and the formulae of 1st year are valid.

(II) If $f : U \to \mathbb{R}^m$, $g : V \to \mathbb{R}^k$, $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open, $f(x_0) \in V$, and $f'(x_0)$, $g'(f(x_0))$ exists, then so does $(g \circ f)'(x_0)$, and $(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0)$ (composition of linear maps (mult^n if n)

$\left(: \mathbb{R}^n \xrightarrow{f'(x_0)} \mathbb{R}^m \xrightarrow{g'(f(x_0))} \mathbb{R}^k \right)$

(6.8) III Suppose $f : [a,b] \to [c,d]$ is a bijection, is differentiable at $x_0 \in [a,b]$, (m=n=1) and $f'(x_0) \neq 0$. Then the inverse function $f^{-1} : [c,d] \to [a,b]$ is differentiable at $f(x_0)$ and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

$\left(m = n > 1 \;:\; \ldots \;\; \frac{1}{f'(x_0)} \text{ mean } f'(x_0)^{-1} \right)$

DaP

Pf. (I): See the book for 2 different pfs of the differentiability of fg. The second pf also works for the case where (e.g.) $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ so that $fg : \mathbb{R}^n \to \mathbb{R}^m$. Then $(fg)'(x_0)(v) = f'(x_0)(v) \cdot g'(x_0)(v)$, $\forall v \in \mathbb{R}^n$. (e.g in alg d)

(II) Write
$$f(v) = f(v_0) + L(v - v_0) + r(v)\,\|v - v_0\|$$
$$g(w) = g(w_0) + M(w - w_0) + s(w)\,\|w - w_0\|,$$
where $r : U \to \mathbb{R}^m$, $s : V \to \mathbb{R}^k$, $M : \mathbb{R}^m \to \mathbb{R}^k$ & $L : \mathbb{R}^n \to \mathbb{R}^m$ are linear,

$w_0 = f(v_0)$, r, s cts at v_0, w_0 resp & $r(v_0) = 0$, $s(w_0) = 0$.

Then, $(w = f(v))$
$$g \circ f(v) = g(f(v_0)) + M(f(v) - f(v_0)) + s(f(v))\,\|f(v) - f(v_0)\|$$
$$= g(f(v_0)) + M(L(v_0 - v_0)) + \|v - v_0\|\, M(r(v))$$
$$+ s(f(v))\,\|L(v - v_0) + r(v)\,\|v - v_0\|\,\|.$$

Now, define
$$t(v) = \begin{cases} 0, & v = v_0 \\ s(f(v))\left\| \frac{L(v - v_0)}{\|v - v_0\|} + r(v) \right\| + M(r(v)), & v \neq v_0 \end{cases}$$

Then $g \circ f(v) - g \circ f(v_0) = M \circ L(v - v_0) + t(v)\,\|v - v_0\|$. It remains to show that t is cts at v_0.

Since r is cts at v_0, $r(v_0)=0$, and M is cts at 0 & $M(0)=0$, $\lim_{v\to v_0} M(r(v))=0$. (55)

Moreover, f diff'ble at $v_0 \Rightarrow f$ cts there, and so $s\circ f$ is too, and $\lim_{v\to v_0} s(f(v)) = s(f(v_0)) = 0$

Thus, it suffices to show that $\left\| \frac{L(v-v_0)}{\|v-v_0\|} + r(v) \right\|$ is bdd near v_0. But $\lim_{v\to v_0} r(v)=0$,

and so it suffices to show that $\frac{\|L(v-v_0)\|}{\|v-v_0\|}$ is bdd. But L is cts, and from

ass 4 Q4, we know then that $\exists K$ s.t. $\|L(v-v_0)\| \le K\|v-v_0\|$. Hence $\lim_{v\to v_0} t(v) = 0 = t(v_0)$

Thus $(g\circ f)'(v_0)$ exists and equals $M\circ L = g'(f(x_0))\circ f'(x_0)$.

(II) Outline only $f(x)-f(x_0) = L(x-x_0) + r(x)\|x-x_0\|$, and L is invertible, so with $y=f(x)$, $y_0=f(x_0)$, $x=f^{-1}(y)$, $x_0=f^{-1}(y_0)$, we have

$$L^{-1}(y-y_0) = x-x_0 + L^{-1}(r(x)).\|x-x_0\|, \text{ or}$$

$$y\neq y_0 \quad f^{-1}(y)-f^{-1}(y_0) = L^{-1}(y-y_0) - L^{-1}(r(f^{-1}(y)))\frac{\|x-x_0\|}{\|y-y_0\|}.\|y-y_0\|$$

It suffices to show that $\frac{\|x-x_0\|}{\|y-y_0\|}$ is bdd (above) near x_0. With $h=x-x_0$,

this is $\frac{\|h\|}{\|L(h)+r(x_0+h)\|h\|\|} = A$. Since L is invertible, $\overset{(ex.)}{\exists}\, c>0$ s.t. $\|L(h)\| \ge c\|h\|$,

and $\exists\,\delta>0$ s.t. $\|h\|<\delta \Rightarrow \|r(x_0+h)\| < \frac{c}{2}$, so $\|L(h)+r(x_0+h)\|h\|\| \ge \|L(h)\| - \|r(x_0+h)\|\|h\|$

$> c\|h\| - \frac{c}{2}\|h\| \ge \frac{c}{2}\|h\|$. Hence $A < \frac{2}{c}$ is bdd. Note: $(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}$ ■