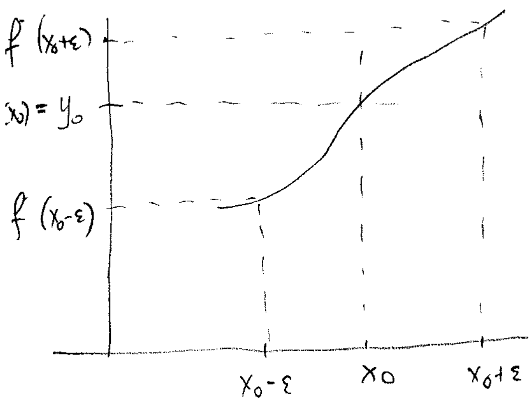
We now know that if $f:[a,b) \to \mathbb{R}$ is cts, $\exists\, c,d \in \mathbb{R}$ s.t
$f:[a,b) \to [c,d]$ is a bijection!

∴ **ex.** If f as above is increasing and bijective, so is $f^{-1}:[c,d] \to [a,b]$.

Theorem: Suppose $f:[a,b) \to [c,d]$ is a cts bijection. Then, $f^{-1}:[c,d] \to [a,b]$ is also cts.

Pf. Wlog. assume that f is increasing (so f^{-1} is, as well, by the exercise.)



Let $y_0 \in [c,d]$, $x_0 = f^{-1}(y_0)$ and $\varepsilon > 0$.

Let $\delta = \min\big(f(x_0+\varepsilon)-f(x_0),\ f(x_0)-f(x_0-\varepsilon)\big)$

∴ $\delta \le f(x_0+\varepsilon)-f(x_0)$, (*) $\delta \le f(x_0)-f(x_0-\varepsilon)$. (**)

Then if y satisfies $|y-y_0|<\delta$, then
$$f(x_0)-\delta < y < f(x_0)+\delta$$

But $f(x_0-\varepsilon) \underset{(\text{by } (**))}{\le} f(x_0)-\delta$ and $f(x_0)+\delta \underset{(\text{by } (*))}{\le} f(x_0+\varepsilon)$

Hence $|y-y_0|<\delta \Rightarrow f(x_0-\varepsilon) < y < f(x_0+\varepsilon) \Rightarrow x_0-\varepsilon < f^{-1}(y) < x_0+\varepsilon$

ie. $|y-y_0|<\delta \Rightarrow |f^{-1}(y)-f^{-1}(y_0)|<\varepsilon$. Hence f^{-1} is cts at y_0. ▨

e.g. Let $f:[0,n] \to [0,n^2)$ be $f(x)=x^2$, $(n \in \mathbb{N})$. Then
$f^{-1}:[0,n^2] \to [0,n)$ ie. $\sqrt{\ }:[0,n^2] \to [0,n]$ is also cts.
Since this holds $\forall n \in \mathbb{N}$, $\sqrt{\ }:[0,\infty) \to [0,\infty)$ is cts.

Remark: There is another way to formulate the defn of continuity.

Limits: Let $f:A \to \mathbb{R}^n$, and $l \in \mathbb{R}^n$. Then if f is defined near a,
$$\lim_{x\to a} f(x) = l \quad \text{iff} \quad \forall \varepsilon>0\ \exists \delta>0 \text{ s.t.}$$
$$0<\|x-a\|<\delta \Rightarrow \|f(x)-l\|<\varepsilon.$$

(It is not actually necessary for f to be defined at a.)

ex. f is cts at a if $\lim_{x\to a} f(x) = f(a)$.

e.g. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = 1.$ | $\frac{\sin x}{x}$

Uniform convergence III.4

Idea: have $\{f_n\}_{n\geq 1}$, $f_n : A \to \mathbb{R}$ is a sequence of functions ($A \subseteq \mathbb{R}^m$). Does $\lim\limits_{n\to\infty} f_n$ exist as $f : A \to \mathbb{R}$? What properties will it have? eg f_n cts on A $\forall n \overset{?}{\Rightarrow} \lim\limits_{n\to\infty} f_n$ "

$$f_n' \text{ exists} \overset{?}{\Rightarrow} \left(\lim_{n\to\infty} f_n\right)' \text{ exists} \overset{?}{=} \lim_{n\to\infty} f_n'$$

$$\int_A f_n \text{ exists} \Rightarrow \int_A \lim_{n\to\infty} f_n \text{ exists} \overset{?}{=} \lim \int_A f_n .$$

There are 2 notions of convergence to consider

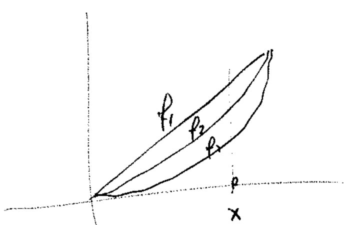
Suppose $f : A \to \mathbb{R}$, $f_n : A \to \mathbb{R}$, $\forall n \geq 1$.

Pointwise Convergence of functions: We say $\lim\limits_{n\to\infty} f_n = f$ pointwise or "$f_n \to f$ pointwise" if, $\forall x \in A$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ (convergence of numerical sequence, one for each point $x \in A$)

converges to

i.e. $\forall x \in A \ \forall \varepsilon > 0 \ \exists N = N(x,\varepsilon)$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

e.g. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$.



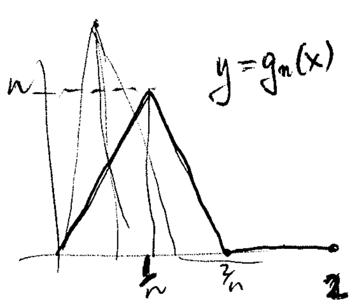
For $x \in [0,1)$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n = 0$
$(0 \leq x < 1)$

But $\lim\limits_{n\to\infty} f_n(1) = 1$

$\therefore$ If we define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$

Then $f_n \to f$ pointwise. Note f_n is cts $\forall n$
f is not.

e.g.



$\underline{n \geq 1}$

$$g_n(x) = \begin{cases} n^2 x, & x \in [0, \frac{1}{n}] \\ n(2 - nx), & x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & x \in [\frac{2}{n}, 2] \end{cases}$$

$\int_0^2 g_n = 1$

$\frac{2}{n} \cdot \frac{1}{2} \cdot n = 1$

$\forall x \in [0,2]$ $\lim\limits_{n\to\infty} g_n(x) = 0$; $g_n \to 0$ (zero fn) ptwise
$g(x) = 0$

$N(x,\varepsilon) \geq \frac{x}{2}$

However $\int_0^2 g_n(x)\,dx = 1$, $\int_0^2 g(x)\,dx = \int_0^2 0\,dx = 0$

Hence $\lim_{n\to\infty} \int_0^2 g_n(x)\,dx \neq \int_0^2 \lim_{n\to\infty} g_n(x)\,dx$.

A stronger notion of convergence : $\{f_n\}_{n\geq 1}$, $f_n : A \to \mathbb{R}$, $f : A \to \mathbb{R}$

Defn 4.1 (W 1841) The sequence $\{f_n\}_{n\geq 1}$ converges uniformly on A to a functn f if "$f_n \to f$ unif on A"
(P213)

$\forall \varepsilon > 0 \quad \exists N$ st $\forall n \geq N$, $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall x \in A$, $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$

Remarks (1) $N = N(\varepsilon)$ only (not $N = N(\varepsilon, x)$.)

Contrast with (fully written out) defn of pointwise convergence on A:

$\forall x \in A \quad \forall \varepsilon > 0 \quad \exists N \ (= N(x,\varepsilon))$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

(e.g. $f_n(x) = x^n$, $N \geq \frac{1-x}{\varepsilon x} \Rightarrow |x^n| < \varepsilon$. ; clear for gn $N = N(\varepsilon, x)$).

(2) $f_n \to f$ uniformly on A iff $a_n = \sup_{x\in A} \| f_n(x) - f(x) \| \to 0$.
$=: \|f_n - f\|$

iff $\|f - f_n\| \to 0$.

Cauchy criterion (3) f_n converges $\to f$ unif. on $A \Leftrightarrow \forall \varepsilon > 0\ \exists N$ s.t. $\forall k \geq 1$, $\forall x \in A$ $\|f_{n+k}(x) - f_n(x)\| < \varepsilon$.
(and $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall x \in A$).

Thm (W, 1861) If $f_n : A \to \mathbb{R}^p$ ($A \subseteq \mathbb{R}^n$) $n \geq 1$ are cts and $f_n \to f$ uniformly on A, then $f : A \to \mathbb{R}^p$ is cts.
Thm 4.2
P213

Pf. Let $a \in A$ and $\varepsilon > 0$. Then $\exists N$ s.t. $\forall x \in A$ $n \geq N \Rightarrow \|f_n(x) - f(x)\| < \frac{\varepsilon}{3}$ $(*)$

Since the single fn f_N is cts at a, $\exists\ \delta > 0$ s.t. $\|x - a\| < \delta \Rightarrow \|f_N(x) - f_N(a)\| < \frac{\varepsilon}{3}$ $(**)$

Hence, if $\|x - a\| < \delta$,

$$\|f(x) - f(a)\| \leq \|f(x) - f_N(x)\| + \|f_N(x) - f_N(a)\| + \|f_N(a) - f(a)\|$$

$$< \frac{\varepsilon}{3}\ (*) + \frac{\varepsilon}{3}\ (**) + \frac{\varepsilon}{3}\ \underset{f_n\to f}{(*)} = \varepsilon. \quad \square$$

Remarks (I) Recall ass#3, Q2; $V = \{ f : A \to \mathbb{R} \mid f \text{ is bdd on } A \}$
If $f \in V$
$\|f\| = \sup \{ |f(x)| \mid x \in A \} = \|f\|_\infty$.

Exercise
Claim If $f_n : A \to \mathbb{R}$ and f_n is bdd on A ($f_n \to f$ uniformly on A) $\iff \lim_{n\to\infty} \|f - f_n\| = 0$.

Pf. Suppose $f_n \to f$ unif on A. Let $\varepsilon > 0$, choose N st. $\forall n \geq N$, $\forall x \in A$,
(f is bdd) $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then $\forall n \geq N$, $\sup \{ |f_n(x) - f(x)| \mid x \in A \} \leq \frac{\varepsilon}{2} < \varepsilon$
Hence i.e. if $a_n = \|f - f_n\|$, then $n \geq N \Rightarrow 0 \leq a_n < \varepsilon$. Hence $\lim_{n\to\infty} \|f - f_n\| = 0$.

Now suppose $\lim_{n\to\infty} \|f - f_n\| = 0$, let $\varepsilon > 0$. Then $\exists N$ st. $\forall n \geq N$,
$\sup \{ |f_n(x) - f(x)| \mid x \in A \} < \varepsilon$. Hence, $\forall n \geq N$, $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$
i.e. $\exists N = N(\varepsilon)$ st $\forall x \in A$, $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$. Hence
$f_n \to f$ unif on A. ∎

(II) $\|f - f_n\|_\infty \to 0$ is called "convergence in norm" ("sup norm" $\| \ \|_\infty$).

(III) Once you've seen integrals,

Thm III.5.19 (2021) If $\forall n$, f_n is cts on $[a,b]$, and $f_n \to f$ uniformly on $[a,b]$, then
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx = \int_a^b f(x)\,dx.$$

Pf (outline only;) We note that f_n, f are cts and $\therefore$ "integrable".
Let $\varepsilon > 0$. Then $\exists N$ st. $\forall n \geq N$, $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{(b-a)}$.

$$\text{Then } \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \underset{\text{linearity of } \int_a^b \_\,dx}{=} \left| \int_a^b (f_n - f)(x)\,dx \right| \underset{\substack{\text{property of integral}\\ \text{(to be seen)}}}{\leq} \int_a^b |f_n(x) - f(x)|\,dx$$

"$\int_a^b$" is monotonic

$$\underset{\text{cts fns}}{<} \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon.$$ Hence $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. ∎

L10 L11 **Important examples** Suppose $f_i : A \to \mathbb{R}$, $i \in \mathbb{N}$

and set $S_n(x) = \sum_{i=0}^{n} f_i(x)$.

Def^n

(I) $\sum_{i=0}^{\infty} f_i(x) = f(x)$ on A iff $\forall x \in A$ $\lim_{n\to\infty} S_n(x) = f(x)$. i.e. $S_n \to f$ pointwise.

"$\sum f_i(x)$ converges (ptwise) on A"

(II) We say $\sum_{i=0}^{\infty} f_i(x)$ is uniformly convergent (to $f(x)$) on A ("converges uniformly")

if $S_n \to f$ uniformly on A.

"W-M test"

Thm (III.4.3) (W) Suppose (a) $|f_n(x)| \le M_n$, $\forall x \in A, \forall n \in \mathbb{N}$ and that

(b) $\sum_{n\in\mathbb{N}} M_n$ converges. Then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on A.

N.B.

Pf. Let $\varepsilon > 0$ and choose $N (= N(\varepsilon))$ s.t. $\forall n \ge N$, $\forall k \ge 1$

$0 \le M_{n+1} + \cdots + M_{n+k} < \varepsilon$.

Then, $\forall n \ge N$, $\forall k \ge 1$, $\forall x \in A$ $|f_{n+1}(x) + \cdots + f_{n+k}(x)|$

$\le |f_{n+1}(x)| + \cdots + |f_{n+k}(x)| \le M_{n+1} + \cdots + M_{n+k} < \varepsilon$. Let $S_n(x) := \sum_{i=0}^{n} f_i(x)$. Then,

$\forall n \ge N$, $\forall k \ge 1$, $\forall x \in A$ $|S_{n+k}(x) - S_n(x)| < \varepsilon$. $\therefore \lim_{n\to\infty} S(x) =: f(x)$ exists, $\forall x \in A$,

and $S_n \to f$ uniformly on A (Cauchy criterion). ∎

e.g. $\sum_{n=0}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly on $\mathbb{R}$, since, $\forall x \in \mathbb{R}$

$\left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2}$

and $\sum \frac{1}{n^2}$ converges!

e.g. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a cts fn of x! On $[-K, K]$, $|S_k(x)| = \left|\sum_{n=0}^{k} \frac{x^n}{n!}\right| \le \sum_{n=0} \frac{K^n}{n!}$ $\left|\frac{x^n}{n!}\right| \le \frac{K^n}{n!}$

let $K \in \mathbb{N}$

On $[-K, K]$, $\left|\frac{x^n}{n!}\right| \leq \frac{m^n}{n!}$. We know $\sum_{n=0}^{\infty} \frac{K^n}{n!}$ converges (49)

(e.g... ratio test $\frac{K^{n+1}}{K^n} \cdot \frac{n!}{(n+1)!} = \frac{K}{n} < 1$ for $n > K$), so by the Weierstrass M test,

~~So does~~ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-K, K]$ to a function $f(x) =$

e^x. Now f is cts, because each of the functions $S_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$;

being polynomials, are, and $S_n \to f$ uniformly!

Hence $f(x) = e^x$ is cts on $[-K, K]$ $\forall K \in \mathbb{N}$, and

so $f(x) = e^x$ is cts on $\mathbb{R}$.

Rmk (1) (for future use): $\tilde{S}_n(x) := \sum_{j=1}^{n} \frac{x^{j-1}}{(j-1)!}$ converges uniformly to a cts fn on $[-K, K]$, by the same argument. Indeed, note that

$\tilde{S}_n(x) = S_{n-1}(x)$, and so $\lim_{n\to\infty} \tilde{S}_n(x) = \lim_{n\to\infty} S_n(x) = e^x$, for all $x \in \mathbb{R}$.

(2) Note that $x < y \Rightarrow S_n(y) - S_n(x) > y - x$. Hence $\lim_{n\to\infty}(S_n(y) - S_n(x)) \geq y - x > 0$ if $x < y$. i.e. $e^y - e^x > 0$ if $x < y$ (injective). Thus $x \mapsto e^x$ is strictly increasing. Moreover, since $e^x = \lim_{n\to\infty} S_n(x) \geq 1 + x$, $\forall x$, given any $y \in \mathbb{R}$, $y \geq 1$,

$\exists x_0$ s.t. $e^{x_0} > y$. Now, $e^0 = 1 \leq y$ and $\exp: [0, x_0] \to \mathbb{R}$ is cts, so

by IVT, $\exists x \in [0, x_0]$ s.t. $e^x = y$! Hence $x \mapsto e^x$ is surjective, as a map

$\exp: [0, \infty) \to [1, \infty)$.

Def^n $\log: [1, \infty) \to [0, \infty)$ is the inverse f^n of $\exp$.