Properties of cts fn.

<u>Theorem 3.4</u> Suppose $f, g : A \to \mathbb{R}$ are both cts at $a \in A$.

Then, 1) $f+g$ is cts at a.
2) $f \cdot g$ is cts at a.
3) If $g(a) \neq 0$, $\frac{1}{g}$ (is defined near a and) is cts at a.

Pf. 1) Let $\{a_n\}_{n \geq 1} \subseteq A$ be any sequence with $a_n \to a$. To show $f+g$ is cts at a, it suffices to show that $(f+g)(a_n) \longrightarrow (f+g)(a)$. But this is the same as showing that $(f(a_n) + g(a_n)) \longrightarrow f(a) + g(a)$. Since f, g are cts at a, $f(a_n) \to f(a)$ & $g(a_n) \to g(a)$. A previous theorem allows us to conclude!

2) Argue with sequences $a_n \to a$ and use the corresponding theorem for limits of products of sequences.

3) First we show $\exists \delta > 0$ s.t. $g(x) \neq 0$ if $|x-a| < \delta$, under the assumption $g(a) \neq 0$. Let $\varepsilon = \frac{|g(a)|}{2} > 0$, and choose $\delta > 0$ s.t. $|x-a| < \delta \Rightarrow |g(x) - g(a)| < \varepsilon$.
Then, $|g(a)| = |g(a) - g(x) + g(x)| \leq |g(a) - g(x)| + |g(x)|$ so $|g(x)| \geq |g(a)| - |g(a) - g(x)|$ $> |g(a)| - \varepsilon$ if $|x-a| < \delta$. But $|g(a)| - \varepsilon = \frac{|g(a)|}{2} > 0$. So $|x-a| < \delta \Rightarrow |g(x)| > \frac{|g(a)|}{2} > 0$.

Now let $a_n \to a$ and argue as in the proof that $a_n \to a \neq 0 \Rightarrow \frac{1}{a_n} \to \frac{1}{a}$ (using n large enough so $a_n \neq 0$).

<u>Remark</u> (A) (1) can be stated and the proof is identical if $A \subseteq \mathbb{R}^n$ & $f, g : A \to \mathbb{R}^n$.
(B) If $A \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}^m$, the statement (2) is still valid and the proof is (almost) identical.
(C) $A \subseteq \mathbb{R}^n$, $g : A \to \mathbb{R}$ still OK.

<u>Corollary</u> ① f, g cts at $a \in A$, $g(a) \neq 0 \Rightarrow f/g$ is defined "near a" and is cts at a.
② f cts at $a \Rightarrow f$ is bounded "near a".

<u>Remark</u> "near a" can be made precise in the following way:

A function f has a property "near a" if $\exists \delta > 0$ s.t. $\forall x \in B(a, \delta)$, $f(x)$ has the property.

i.e. $\exists \delta > 0$, $|x-a| < \delta \Rightarrow f(x)$ has the property.

Remarks on sequences in $\mathbb{R}^p$.

In last Friday's D&D, you saw, if $v_n = ($

$\{v_n\}_{n\geq 1} \subset \mathbb{R}^p$ is Cauchy in $\mathbb{R}^p$ $\Leftrightarrow$ Each of the component sequences is Cauchy in $\mathbb{R}$.

e.g. $p=2$ $\{(x_n, y_n)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}^2$ $\Leftrightarrow$ $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are Cauchy in $\mathbb{R}$.

Since we know Cauchy sequences in $\mathbb{R}$ converge, $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ Cauchy $\Rightarrow \exists x, y \in \mathbb{R}$ s.t. $x_n \to x$, $y_n \to y$. Then (using the inequality $\|(a, b)\| \leq \sqrt{2} \max\{|a|, |b|\}$) $(x_n, y_n) \to (x, y)$. Thus,

Theorem A sequence is Cauchy in $\mathbb{R}^p$ iff it converges.

Recall: Bolzano-Weierstrass: Every bdd sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem (L11/L12) B-W for $\mathbb{R}^p$: Every bounded sequence in $\mathbb{R}^p$ has a convergent subsequence:

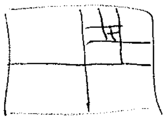
Pf (i) ($p=2$) Suppose $v_n = (x_n, y_n)$, $n \geq 1$, $L>0$, and $\|v_n\| \leq L$, $\forall n \geq 1$. Then, $|x_n| \leq \|v_n\| \leq L$ and $|y_n| \leq \|v_n\| \leq L$, $\forall n \geq 1$, so both $\{x_n\}_{n\geq 1}$ & $\{y_n\}_{n\geq 1}$ are bdd. Apply B-W ($p=1$) to $\{x_n\}_{n\geq 1}$ to obtain a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ with $\lim_{k\to\infty} x_{n_k} = a$. To avoid confusing notation, rename this by defining $a_k = x_{n_k}$. So $a_k \to a$. Now define $b_k = y_{n_k}$, $k \geq 1$. Clearly, $\forall k$, $|b_k| \leq L$ because $\{y_n\}_{n\geq 1}$ is bdd by L. Hence, using B-W($p=1$) again, there is a convergent subsequence $\{b_{k_\ell}\}_{\ell\geq 1}$. That is, $\exists b \in \mathbb{R}$ s.t. $\lim_{\ell\to\infty} b_{k_\ell} = b$. Since $\{a_k\}_{k\geq 1}$ converges, every subsequence also converges and so $\{a_{k_\ell}\}_{\ell\geq 1}$ converges, and to a. That is, $\lim_{\ell\to\infty} a_{k_\ell} = a$.

Now let $\tilde{v}_\ell = (a_{k_\ell}, b_{k_\ell}) \; (= (x_{n_{k_\ell}}, y_{n_{k_\ell}}))$, $\ell \geq 1$.

Then $\{\tilde{v}_\ell\}_{\ell\geq 1}$ is a subsequence of $\{v_n\}_{n\geq 1}$ and

$$\lim_{\ell\to\infty} \tilde{v}_\ell = \lim_{\ell\to\infty} (a_{k_\ell}, b_{k_\ell}) = \left(\lim_{\ell\to\infty} a_{k_\ell}, \lim_{\ell\to\infty} b_{k_\ell}\right) = (a,b).$$

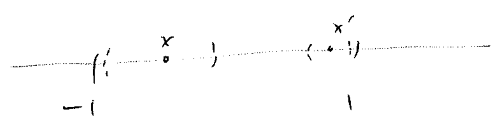
Hence $\{\tilde{v}_\ell\}$ is a convergent subsequence of $\{v_n\}_{n\geq 1}$. ▨

Pf (2) ...  ... | Use bisection, just as in $p=1$.

Recall (Ass #3, Q2) If $\{c_n\}_{n\geq 1} \subset [a,b]$, and $c_n \to c$, then $c \in [a,b]$. We know that if the interval is not "closed", this is no longer true: eg $\{\frac{1}{n}\}_{n\geq 1} \subset (0,1]$, $\frac{1}{n} \to 0 \notin (0,1]$.

What is the higher dimensional analogue of "closed"?
"complement"

Def^n 1) $U \subseteq \mathbb{R}^n$ is an <u>open</u> set if $\forall x \in U, \exists r > 0$ s.t. $B(x,r) \subseteq U$.



2) $F \subset \mathbb{R}^n$ is a <u>closed</u> set if $\mathbb{R}^n \setminus F = \mathbb{R}^n - F$ is open

i.e. if F^c is open.

e.g. $n=1$; • open intervals are open.

Then $B(x,r) \subseteq (a,b)$

• open balls in $\mathbb{R}^n$ are <u>open</u> (prov. ex; Ass #3, Q6)

ex. • a union of any collection of open sets is open
• an intersection of a <u>finite</u> number of open sets is open
N.B. e.g. $\bigcap_{n\geq} (-\frac{1}{n}, 1) \overset{ex.}{=} [0,1)$ is <u>not</u> open

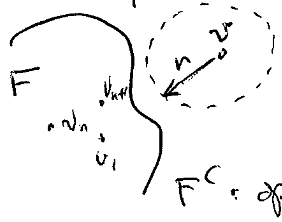
• closed intervals are closed ; $[a,b] = (-\infty, a) \cup (b, \infty)$ (open, open)
• finite* unions of closed intervals $\bigcup_{n\geq 1} [0, 1-\frac{1}{n}] = [0,1)$ is not open!
• arbitrary intersect^n of closed sets are closed

Theorem. Suppose F is a closed set in $\mathbb{R}^p$ and $\{v_n\}_{n\geq 1}$ is a sequence s.t 1) $\{v_n\}_{n\geq 1} \subseteq F$

& 2) $\exists v \in \mathbb{R}^n$ st $v_n \to v$. (i.e. $\{v_n\}_{n\geq 1}$ Converges!)

Then, $v \in F$.

Pf. We proceed by contradiction: Suppose $v \notin F$. Then $v \in F^c = \mathbb{R}^n \setminus F$.



But F^c is open, and $v \in F^c$, so $\exists r > 0$ s.t. $B(v, r) \subseteq F^c$. Now let $\varepsilon = r > 0$ Then $\exists N$ s.t. $\forall n \geq N$, $\| v - v_n \| < \varepsilon$. That is, $n \geq N \Rightarrow v_n \in B(v, \varepsilon) = B(v, r) \subseteq F^c$. In particular, $v_N \in F^c$ so $v_N \notin F$. This is a contradiction. Hence $v \in F$. ▨

Remark: We'll see in a moment F is closed in $\mathbb{R}^p \iff$ Convergent sequences from F have their limits in F

Suppose F is closed & bounded & and $\{v_n\}_{n\geq 1} \subset F$. Then, by B-W, there is a convergent subsequence $\{v_{n_k}\}_{k\geq 1}$, and by the previous theorem $\lim_{k\to\infty} v_{n_k} \in F$.

Def^n (Fréchet, 1906) (P. 281) A subset $K \subset \mathbb{R}^p$ is "sequentially compact" compact if every sequence in K has a convergent subsequence, with its limit in K. (Rmk. "Compact" : generalization of "finite").

Goal: K compact, f cts on $K \Rightarrow f(K)$ compact

Thm $K \subseteq \mathbb{R}^p$ is compact $\iff$ K is closed and bounded.

Pf. "$\Leftarrow$" is already done! ("Bdd" assures via B-W, a convergent subsequence, and closed assures limit $\in K$.)

"$\Rightarrow$": We proceed by proving the contrapositive, namely "K not closed or K not bdd $\Rightarrow$ K not compact". If K is not closed, then $U = \mathbb{R}^p \setminus K$ is not open. So $\exists v \in U$ s.t. $\forall r > 0$, $B(v, r) \not\subseteq U$. Hence for any n, $r = \frac{1}{n} > 0$, $\exists v_n \in B(v, \frac{1}{n})$ & $v_n \notin U$. Then, $v_n \in K$, $\forall n$ so $\{v_n\}_{n\geq 1} \subseteq K$, but $v_n \to v \notin K$. Hence K cannot be compact.

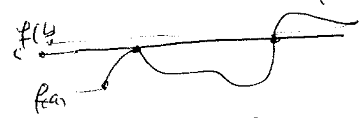
If K is not bounded, $\forall n \in \mathbb{N}$, $\exists v_n \in K$ s.t. $\|v_n\| \geq n$. Since $\{v_n\}_{n\geq 1}$ and hence all its subsequences are not bdd, so none can converge. Hence K is not compact ▨

e.g. every finite set in $\mathbb{R}^p$ is compact (ex.) $A = \{\frac{1}{n} \mid n \geq 1\}$ is not compact.
e.g. A "closed" box $B = \{(x_1, \ldots, x_p) \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq p\}$ is compact (ex: a closed box is closed & bdd.

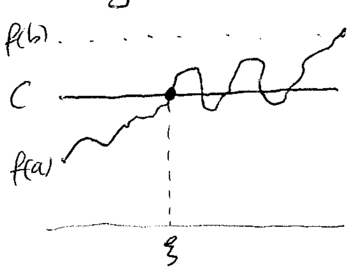
<u>Thm</u> If $f : K \longrightarrow \mathbb{R}^m$ is cts (on K) and K ($\subset \mathbb{R}^p$) is compact, $f(K)$ is compact in $\mathbb{R}^m$!
Pf. Let $\{v_n\}_{n \geq 1} \subset f(K)$ be a sequence in $f(K)$. Then, $\forall n$, $v_n = f(w_n)$ for some $w_n \in K$. So $\{w_n\}_{n \geq 1} \subset K$. But every sequence in K has a convergent subsequence that converges to something in K, so let $\{w_{n_k}\}_{k \geq 1}$ be such that $\lim_{k \to \infty} w_{n_k} = w \in K$.
Now consider the sequence $\{v_{n_k}\}_{k \geq 1} = \{f(w_{n_k})\}_{k \geq 1}$. Claim: $\lim_{k \to \infty} v_{n_k} = f(w)$.
i.e. $\lim_{k \to \infty} f(w_{n_k}) = f(w)$. But this is true, since $w_{n_k} \to w$ & f is cts! ▨

<u>Corollary</u> If $[a, b] \subset \mathbb{R}$ is a closed and bounded interval, and $f : [a, b] \to \mathbb{R}$ is cts on $[a, b]$, $f([a, b])$ is closed and bounded!
Even better: we'll see soon that $f([a, b]) = [m, M]$ is a closed interval, where m = min. val of f on $[a, b]$, M = max. value of f on $[a, b]$!

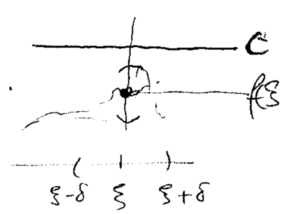
<u>Intermediate Value Theorem</u> (3.8) (Bolzano, 1817). Let $f : [a, b] \to \mathbb{R}$ be cts (on $[a, b]$). Then for every c between $f(a)$ & $f(b)$, $\exists\, \xi \in [a, b]$ s.t. $f(\xi) = c$ PICTURE:



Pf Wlog assume $f(a) < f(b)$, and that $f(a) < c < f(b)$ (otherwise there's nothing to do: take $\xi = a$ or b ...). We actually find the first (smallest) ξ!

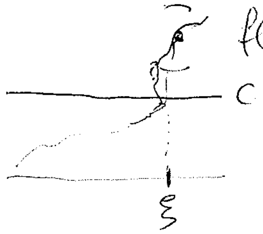


Let $X = \{ s \in [a, b] \mid \forall x \in [a, s],\ f(x) < c \}$.
Then ① X is bdd above by b! ($s \leq b$).
② $X \neq \emptyset$, because $a \in X$!
Hence $\xi = \sup X$ exists. We show $f(\xi) = c$.



Suppose $f(\xi) < c$. With $\varepsilon = \frac{c - f(\xi)}{2}$, $\exists\, \delta > 0$ s.t. $\forall x \in (\xi - \delta, \xi + \delta)$,
$f(\xi) - \varepsilon < f(x) < f(\xi) + \varepsilon$. But, then, $\forall x, |x - \xi| < \delta$, $f(x) < f(\xi) + \varepsilon = \frac{f(\xi) + c}{2} < c$.
This contradicts the def^n of $\xi = \sup X$, since then $f(x) < c$ $\forall x \in [a, \xi] \cup [\xi, \xi + \frac{\delta}{2}]$,
so $\xi + \frac{\delta}{2} \in X$. Hence $f(\xi) \geq c$. A similar argument shows $f(\xi) \leq c$. ▨

(Here it is, for those who've looked: Suppose $f(\xi) > c$.



with $\varepsilon' = \frac{f(\xi)-c}{2}$, $\exists\, \delta' > 0$ s.t.

$x \in (\xi - \delta', \xi + \delta') \Rightarrow f(x) > f(\xi) - \varepsilon = \frac{c + f(\xi)}{2} > c$

But then, $\exists\, s \in X$ with $\xi - \delta' < s \le \xi$, and so $f(s) < c$

and $f(s) > c$. Thus, $f(\xi) \le c$.)

Corollary ① There is a sol^n to $x^5 - 15x - 1 = 0$ in $[0,2]$: let $g(x) = x^5 - 15x - 1$. Then g is cts on $[0,2]$, $-1 = g(0) < 0 < g(2) = 1$ and so $\exists\, \xi \in [0,2]$ (actually $(0,2)$!) s.t. $g(\xi) = 0$.

Corollary ② Let $n \in \mathbb{N}$, $n \ge 2$. $\forall \alpha \in \mathbb{R}$, $\alpha \ge 0$, $\exists\, \beta \in \mathbb{R}$, $\beta \ge 0$ s.t. $\beta^n = \alpha$ (i.e. $\forall \alpha \in \mathbb{R}^+$, $\sqrt[n]{\alpha}$ exists!)

Pf. If $\alpha = 0$, $\beta = 0$ works; if $\alpha = 1$, $\beta = 1$ works. So look at the 2 case $0 < \alpha < 1$, $1 < \alpha$. Consider the function $f(x) = x^n - \alpha$. Note that f is cts on $[0, \alpha]$, and $f(0) = -\alpha < 0$.

If $0 < \alpha < 1$, $f(\alpha) = \alpha^n - \alpha < 0$, so $f(\alpha) < 0 < f(1)$. Hence $\exists\, \beta \in [\alpha, 1]$ s.t. $f(\beta) = 0$. That is, $\beta^n = \alpha$.

If $1 < \alpha$, $f(\alpha) = \alpha^n - \alpha > 0$, so $f(0) < 0 < f(\alpha)$. Hence $\exists\, \beta \in [0, \alpha]$.

Corollary ③ $\forall r \in \mathbb{Q}$, $\forall \alpha \in \mathbb{R}^+$, $\sqrt[r]{\alpha}$ exists. (If $r < 0$, $\sqrt[r]{\alpha}$ exists $\forall \alpha \in \mathbb{R}$.)

The Maximum Theorem (3.9) (W 1861) If $f : [a,b] \to \mathbb{R}$ is cts,

then 1) f is bounded on $[a,b]$

2) f attains its maximum & minimum values i.e. $\exists\, c_1, c_2 \in [a,b]$ s.t. $\forall x \in [a,b]$, $f(c_1) \le f(x) \le f(c_2)$.

Pf 1) is done, since $[a,b]$ is compact & f is cts. Now let $M = \sup\{f(x) \mid x \in [a,b]\}$ (M exists because $f([a,b]) \neq \emptyset$, f bdd above on $[a,b]$). For $\varepsilon = \frac{1}{n}$, choose $x_n \in [a,b]$ s.t. $M - \frac{1}{n} < f(x_n) \le M$. Since $\{x_n\}_{n \ge 1} \subset [a,b]$, & $[a,b]$ is compact, $\exists$ subsequ. $\{x_{n_k}\}_{k \ge 1}$ s.t. $x_{n_k} \to c_2 \in [a,b]$. Then, $f(c_2) = f(\lim_{k\to\infty} x_{n_k}) \overset{f \text{ is cts}}{=} \lim_{k\to\infty} f(x_{n_k}) = M$.

A similar argument shows $\exists\, c_1$ s.t. $f(c_1) = \inf\{f(x) \mid x \in [a,b]\}$. ∎

Corollary ①: If $f : [a,b] \to \mathbb{R}$ is cts $f([a,b]) = [c,d]$ (43)
for some real $c,d \in \mathbb{R}$. (Pf: let c = min. value of f on $[a,b]$,
d = max " ,

and use the IVT.).

Corollary (of ② pf). If $K \subset \mathbb{R}^m$ is compact, and $f : K \to \mathbb{R}$ is cts,
1) $\exists\, m \le M \in \mathbb{R}$ s.t. $\forall\, k \in K$, $m \le f(k) \le M$
2) $\exists\, k_1, k_2 \in M$ s.t. $\forall\, k \in K$, $f(k_1) \le f(k) \le f(k_2)$.

Remarks I) f must be cts at all points of the compact set:
eg $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$ is not cts at the single point $x = 0$
(f is not bdd above on $[0,1]$, attain max)

II) The set must be compact : $\frac{1}{x}$ on $(0,1]$, $[-1,0)$.
x on $[0,\infty)$ (closed set!)

Inverse functions ($\mathbb{R} \to \mathbb{R}$) & their continuity:

Lemma If $f : [a,b] \to \mathbb{R}$ is continuous and injective,
then either a) $\forall\, x_1 < x_2$ in $[a,b]$, $f(x_1) < f(x_2)$ "f is strictly increasing"
or b) $\forall\, x_1 < x_2$ in $[a,b]$, $f(x_1) > f(x_2)$ "f is strictly decreasing"

Pf. Suppose wlog $f(a) < f(b)$. We show f is strictly increasing.
So let $x_1, x_2 \in [a,b]$ with $a < x_1 < x_2 < b$, and define
$g : [0,1] \to \mathbb{R}$ by $g(t) = f((1-t)b + tb_1) - f((1-t)a + ta_1)$.
Note $\tilde{a}(t) = (1-t)a + ta_1$ is cts and $\tilde{a} : [0,1) \to [a, a_1]$
$\tilde{b}(t) = (1-t)b + tb_1$ " " $\tilde{b} : [0,1] \to [b_1, b]$

Then $g(0) = f(b) - f(a) > 0$ while $g(1) = f(b_1) - f(a_1)$.
If $g(1) < 0$ (it can't be 0 because f is injective), $\exists\, t_0 \in (0,1)$ with
$g(t_0) = 0$, i.e. $f(\tilde{a}(t_0)) = f(\tilde{b}(t_0))$. But $\underset{[a,a_1]}{\tilde{a}(t_0)} \neq \underset{[b_1,b]}{\tilde{b}(t_0)}$ C!