Hence $\{[s_{2n+1}, s_{2n+2}]\}_{\geq 1}$ is a nested sequence of intervals, and $|s_{2n+2} - s_{2n+1}| = a_{2n+2} \to 0$, hence, $\exists s \in \mathbb{R}$ st

$$\{s\} = \bigcap_{\geq 1} [s_{2n+1}, s_{2n+2}] \quad \text{i.e.} \quad s = \lim_{n\to\infty} s_n \quad \therefore \sum_{n\geq 0} (-1)^n a_n$$

converges. In particular $\forall k$, $s \in [s_{k-1}, s_k]$, so $|s - s_k| \leq a_k$ ∎

~~**Remark** If a series is convergent, but is not absolutely convergent, the series of +ve terms and the series of -ve terms both diverge. Indeed,~~ $\cancel{\sum_{b_k \geq 0} b_k \to \infty}$

$\cancel{\sum_{b_j \leq 0} b_j \to -\infty.}$

~~Pf: Use notation before~~

$$\cancel{\overset{\sum a_n}{S_n^{(I)}} = S_n^{III} - S_n^{IV}} \;;\quad \cancel{\overset{\sum |a_n|}{S_n^{II}} = S_n^{III} + S_n^{IV}}.$$

$\cancel{S_n^{(I)} \to s}$; $\cancel{S_n^{II} \to \infty}$ ~~(since, can't be bdd, otherwise wd conv.).~~

$$\cancel{S_n^{III} = \tfrac{1}{2}(S_n^{I} + S_n^{II}) \to \infty}$$

$$\cancel{S_n^{IV} = \tfrac{1}{2}(S_n^{II} - S_n^{I}) \to \infty}$$

∎

L9 or L10 **Theorem** a) (Dirichlet, 1837) If a series is absolutely convergent, then the series formed by any rearrangement of its terms converges, with the same sum.

b) (Riemann, 1854) If a series converges but is not absolutely convergent, then $\forall s \in \mathbb{R}$, $\exists$ a rearrangement of the order of the terms s.t. the resulting series converges to s.

proofs will not given in the ...

e.g. (b) $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - 6 + \cdots$

$$s' = \underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{+\frac{1}{6}} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{+\frac{1}{10}} - \frac{1}{12} + \underbrace{\frac{1}{7} - \frac{1}{14}}_{+\frac{1}{14}} - \frac{1}{16} + \frac{1}{9} \cdots$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14}$$

$$= \frac{1}{2} s$$

Idea for (a): Let $\{s_n\}_{n\geq 1}$ be partial sums for $\sum a_n$

$\{t_n\}_{n\geq 1}$ " $\sum a_{f(n)}$ rearrangem $f: \mathbb{N} \to \mathbb{N}$

Show $s_n - t_n \to 0$ as follows: Let $\varepsilon > 0$. Then (absolute) conv. of $\sum_{n\geq 1} |a_n|$ guarantees $N \in \mathbb{N}$ st $\forall n \geq N$, $\forall k \geq 1$

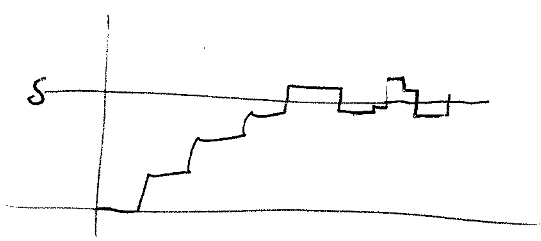
$$|a_{n+1}| + \cdots + |a_{n+k}| < \varepsilon. \quad \cdots * $$

Now choose M s.t. t_M includes all of the terms $\{a_1, \ldots, a_N\}$ (so, $M \geq N$!)

Then, $n \geq M \Rightarrow s_n - t_n = \sum_{j \geq N+1} a_j$, & it's a finite sum.

Use Δ inequality to see $|s_n - t_n| < \varepsilon$ if $n \geq M$.

Idea for (b).



Use +ve terms $\uparrow$ then −ve terms $\downarrow$, alternately; $a_n \to 0$ so rearranged series $\to s$.

Pf. (a) Let $\{a_n\}_{n\geq 1}$ be the sequence of terms & $\{s_n\}_{n\geq 1}$ the sequence of partial sums of $\sum a_m$.

A "rearrangement" of the infinite series $\sum a_n$ involves a bijection $f: \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$. Indeed, let $b_n = a_{f(n)}$, $n \geq 1$; then $\sum_{n\geq 1} b_n$ is a rearrangement of $\sum_{n\in\mathbb{N}} a_m$. (The "rearrangement" depends on & is determ by the bijection f.)

Now let $\{t_n\}_{n\geq 1}$ be the partial sums of $\sum_{n\geq 1} b_n$.

We show that $\forall \varepsilon > 0$, $\exists M$ s.t. $\forall m \geq M$, $|s_m - t_m| < \varepsilon$. Since $s_m \to s$ for some s, this will show that $t_m \to s$ as well.

So let $\varepsilon > 0$. Since $\sum_{n\geq 1} |a_n|$ converges, the Cauchy criterion for convergence guarantees the existence of $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \ \forall k \geq 1, \quad \big||a_{n+1}| + \cdots + |a_{n+k}|\big| = |a_{n+1}| + \cdots + |a_{n+k}| < \varepsilon.$$

Now, we choose $M \in \mathbb{N}$ so that all the terms $a_1, a_2, \ldots, a_N$ appear in the partial sum t_M: indeed, let $M = \max\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(N)\}$. Note that since the numbers $f^{-1}(1), \ldots, f^{-1}(N)$ are all distinct (and all at least 1), $M \geq N$.

Then, if $m \geq M$,

$$s_m = \sum_{i=1}^{N} a_i + \sum_{j=N+1}^{M} a_j \qquad (\text{because } M \geq N)$$

and

$$t_m = \sum_{i=1}^{N} a_i + \sum_{j\in I} a_j, \quad \text{where } I \text{ is finite and, } \min I \geq N.$$

Thus, if $k = \max(I \cup \{M-N\})$,

$$|s_m - t_m| = \Big|\sum_{j=N+1}^{m} a_j - \sum_{j\in I} a_j\Big| \leq |a_{N+1}| + \cdots + |a_{N+k}| < \varepsilon$$

could be some cancellation - we don't care!

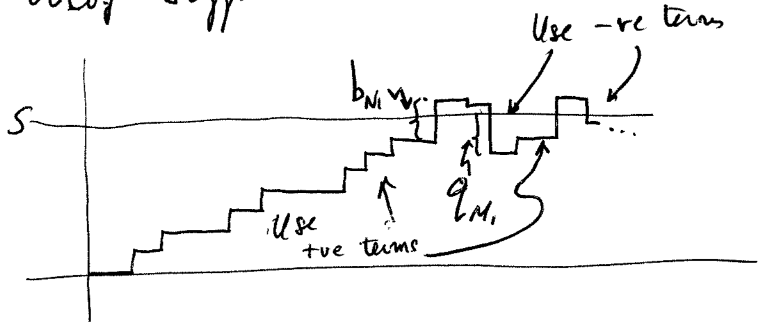
$\boxed{\equiv}$

(b) Let $\sum a_n$ be the series; define DUP

$p_n = \max(0, a_n)$ (≥ 0), $q_n = \min(0, a_n)$ (≤ 0); $\{s_n\}_{n\geq 1}$ partial sums of $\sum a_n$.

From previous remarks, we know that $\sum p_n$ & $\sum q_n$ diverge.

WLOG suppose $S > 0$.



Since $\sum p_n$ is unbdd, let $N_1 = \min\{n \mid \sum_{k=1}^{n} p_k > S.\}$

Thus, $\sigma_1 := s_{N_1} > S$, and so (by defn of N_1) $\sigma_1 - p_{N_1} \leq S$, or

$$b_{N_1} \geq \sigma_1 - S > 0$$

Now define $M_1 = \min\{n \mid S + \sum_{k=1}^{M_1} q_k < S\}$. Then, if

$$\tau_1 := \sigma_1 + \sum_{k=1}^{M_1} q_k \quad , 0 < S - \tau_1 \leq |q_{M_1}|$$

Continue indefinitely in this way: delete from the sequence

$$p_1 \; p_2 \cdots p_{N_1} \; q_1 \; q_2 \; q_3 \cdots q_{M_1} \; p_{N_1+1} \; p_{N_1+2} \cdots p_{N_2}, \cdots$$

all the zeros (this doesn't change the partial sums!) and you obtain a reordering of $\{a_n\}_{n\geq 1}$. Moreover, at the "k^{th} step",

$$0 < \sigma_k - S \leq p_{N_k}$$

$$\text{and } 0 \leq S - \tau_k \leq |q_{M_k}|.$$

Since $a_n \to 0$ (remember: $\sum a_n$ converges!), so do p_{N_k} and q_{M_k} (as $k \to \infty$) and hence the partial sums $\sigma_k, \tau_k \to S$.

III.3 Real functions, Continuity

P.201 **Moral**: a cts f^n is one which preserves convergence of sequences. (i.e. takes Cauchy seq $\to$ Cauchy seq. i.e. $\{a_n\}_{n\geq 1}$ C $\Rightarrow$ $\{f(a_n)\}_{n\geq 1}$ C.)

Set-up: $f : A \longrightarrow \mathbb{R}$ where, at first, $A \subset \mathbb{R}$ and A will be an interval ((a,b), $[a,b]$, (a,∞), etc.) Eventually, we'll consider functions $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ ($n, m \geq 1$). We'll present some notation & def^{ns} (using this notation) that will be used in all dimensions. The def^{ns} will be "dimension invariant".

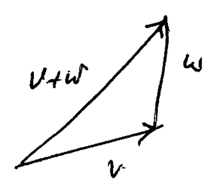
Recall: Euclidean norm of a vector / abs. val of a number / complex mod. of a $z \in \mathbb{C}$ $|\cdot| : \mathbb{R}^n \longrightarrow \mathbb{R}^+ = [0, \infty)$

is def'd by $|\underbrace{(x_1, \dots, x_n)}_{v}| = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{v \cdot v}$

(for $n > 1$, people often use 2 "bars": $\|(x_1, \dots, x_n)\|$. We'll do this when it suits us.)

The Euclidean norm has the following properties:

1) $|v| = 0 \Leftrightarrow v = 0$, $\forall v \in \mathbb{R}^n$
2) $|\lambda v| = |\lambda|\,|v|$, $\forall \lambda \in \mathbb{R}$, $\forall v \in \mathbb{R}^n$
3) $|v + w| \leq |v| + |w|$, $\forall v, w \in \mathbb{R}^n$



Remark. There are norms other than the Euclidean one on $\mathbb{R}^n$, and there are "norms" on vector spaces other than $\mathbb{R}^n$ that satisfy (1)–(3)

exercise: Let $\emptyset \neq A \subset \mathbb{R}$, define $\mathcal{B}(A) = \{ f : A \to \mathbb{R} \mid f \text{ is bounded on } A\}$.

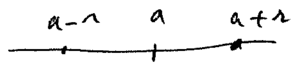
Then (1) $\mathcal{B}(A)$ is a vector space over $\mathbb{R}$ in the 'usual' way

(2) If we define, for $f \in \mathcal{B}(A)$, $|f| = \sup\{f(x) \mid x \in A\}$, then this map $|\,| : \mathcal{B}(A) \to \mathbb{R}^+$ satisfies (1)–(3) above.

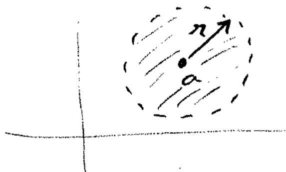
Using the norm, we define the notion of an "open ball"

Def^n Let $a \in \mathbb{R}^n$, $r \in \mathbb{R}$, $r > 0$. The open ball, centre a, radius r is

$$B(a,r) := \{ v \in \mathbb{R}^n \mid |v-a| < r \}.$$

$n=1$ 

, $B(a,r) = (a-r, a+r)$ ← this has been important so far...

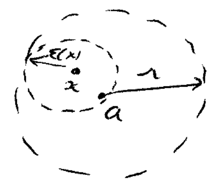
$n=2$ 

Remark ① "Ball" comes from $n=3$.

② "Open" comes from $n=1$ – $B(a,r)$ is what is called an open interval.

Exercise: Prove that $\forall x \in B(a,r)$, $\exists\, \varepsilon = \varepsilon(x) > 0$ s.t.

$$B(x, \varepsilon(x)) \subseteq B(a,r) \ !$$



(Finally!)

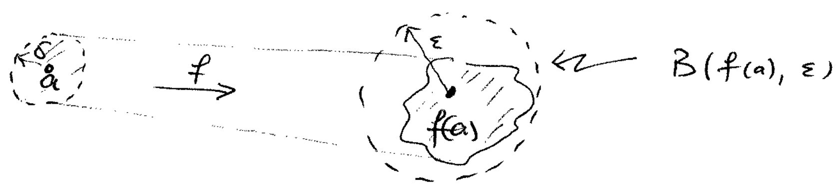
Def^{n}s (Bolzano ~ 1817, Weierstrass ~ 1874) Let $A \subseteq \mathbb{R}^n$, and $a \in A$.

- A function $f : A \longrightarrow \mathbb{R}^m$ is continuous at a if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ \text{s.t.} \ x \in A \text{ and } |x-a| < \delta \Rightarrow |f(x) - f(a)| <$$

i.e. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x \in A \cap B(a,\delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$

picture for $n = m = 2$

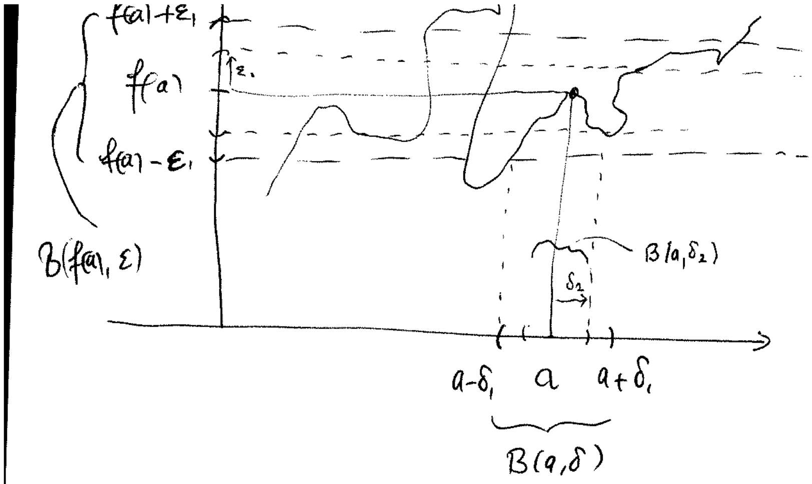


i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(A \cap B(a,\delta)) \subseteq B(f(a), \varepsilon)$.

- $f : A \to \mathbb{R}$ is continuous on A if, $\forall a \in A$, f is cts at a.

Remark: For the moment, we will concentrate on $m = n = 1$.

$n = m = 1$



Remarks • The "δ" will, in general, depend on ε, $\delta = \delta(\varepsilon)$; the smaller the ε, (generally) the smaller δ will have to be.

• The δ will also depend on $a \in A$, as well: see the picture above; so $\delta = \delta(a, \varepsilon)$, in general.

e.g. Constant functions are continuous: Suppose $\forall x \in A$, $c(x) = c \in \mathbb{R}^m$ (fixed c). Then, given $\varepsilon > 0$, choose $\delta =$ (say!) 1. Then, for (fixed) $a \in A$, if $|x-a| < 1$, $|c(x) - c(a)| = |c - c| = 0 < \varepsilon$. Hence c is cts at $a \in A$. (Since a was arbitrary, c is cts on A.

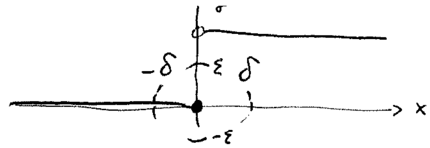
e.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x$, $\forall x \in \mathbb{R}$. Claim: f is cts on $\mathbb{R}$.
Pf Let $a \in \mathbb{R}$, and $\varepsilon > 0$. Then $|f(x) - f(a)| = |3x - 3a| = 3|x-a|$.
Hence, if we choose $\delta = \frac{\varepsilon}{3}$, then $|x-a| < \delta \Rightarrow 3|x-a| < \varepsilon$, i.e. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. (Remark: $f : \mathbb{R}^n \to \mathbb{R}^n$ def'd by $f(v) = cv$, for some fixed $c \in \mathbb{R}$ can be shown to be cts in the same way!)
Hence, f is cts at a. But $a \in \mathbb{R}$ was arbitrary, so f is cts on $\mathbb{R}$. $\boxtimes$
Note: here, $\delta = \delta(\varepsilon)$, but δ did not depend on $a \in \mathbb{R}$.

e.g. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. Then, g is cts on $\mathbb{R}$:
(PROOF) Let $a \in \mathbb{R}$. Then $|g(x) - g(a)| = |x^2 - a^2| = |x+a||x-a|$.
We estimate $|x+a|$ (since "$|x-a|$" is "good") in terms of $|x-a|$ as follows:
$|x+a| = |x - a + 2a| \le |x-a| + 2|a| \le 1 + 2|a|$ as long as $|x-a| \le 1$ (which we will make sure happens! $\therefore$ if $|x-a| \le 1$, $|x^2 - a^2| \le (1+2|a|)|x-a|$. So let $\varepsilon > 0$ and choose $\delta = \min\left(1, \frac{\varepsilon}{1+2|a|}\right)$. Then $|x-a| < \delta \Rightarrow |g(x) - g(a)| < (1+2|a|) \cdot \frac{\varepsilon}{1+2|a|} = \varepsilon$.

to be able to use our estimate $|x-a| < 1$

Here, note that δ depended on both a and ε.

e.g. $f(x) = \begin{cases} 1, & x > 0 \\ 0 & x \le 0 \end{cases}$



Then a) f is cts at a if $a \neq 0$ (Choose $\delta = 1$!!)

b) f is not ct at 0: Note that if "$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$" is false, then "$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x$ with $|x-a| < \delta$ s.t. $|f(x) - f(a)| \geq \varepsilon$"

(i.e. there's a "bad" epsilon for which no δ can guarantee $|x-a|<\delta \Rightarrow |f(x)-f(a)| < \varepsilon$

i.e. " s.t. for every $\delta > 0$, "$\forall x\, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$" fails

i.e. " " $\exists$ "bad" x with $|x-a| < \delta$ s.t. $|f(x) - f(a)| < \varepsilon$ fa

i.e. " " " $|f(x) - f(a)| \geq \varepsilon$.

OK... look at the picture above, and choose $\varepsilon = \frac{1}{2}$. Then, $\forall \delta > 0$, with $x = \frac{\delta}{2}$ we have $|f(x) - f(0)| = 1 > \frac{1}{2} = \varepsilon$. Hence f is not cts at $x = 0$.

exercise $f : [0,1) \to \mathbb{R}$ defi'd by $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1/q, & x = \frac{p}{q} \in \mathbb{Q},\ \gcd(p,q) = 1. \end{cases}$

Show that a) f is cts at a if $a \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$

b) f is not cts at a if $a \in \mathbb{Q} \cap [0,1]$.

$A \subseteq \mathbb{R}$ for the moment.

Thm (3.3) A function $f : A \to \mathbb{R}$ is cts at $a \in A \iff$ for every sequence $\{a_n\}_{n \geq 1} \subseteq A$ with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} f(a_n) = f(a)$.

i.e. $\forall \{a_n\}_{n \geq 1} \subseteq A$, $(a_n \to a) \Rightarrow (f(a_n) \to f(a))$.

Pf. Suppose f is cts at $a \in A$ and $a_n \to a$. Let $\varepsilon > 0$. Since f is cts at a, $\exists\ \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. Since $a_n \to a$, for this $\delta > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a - a_n| < \delta$. But then, if $n \geq N$, $|a - a_n| < \delta$, so $|f(a_n) - f(a)| < \varepsilon$. That is, $n \geq N \Rightarrow |f(a_n) - f(a)| < \varepsilon$. So $f(a_n) \to f(a)$.

Now suppose $f(a_n) \to f(a)$ for all sequence $\{a_n\}_{n \geq 1} \subseteq A$ with $a_n \to a$, but that f is not cts at a. We seek a contradiction, and we do this by finding a sequence $\{a_n\}_{n \geq 1} \subseteq A$ s.t. $a_n \to a$ but $f(a_n) \not\to f(a)$: . Since f is not cts at a, let $\varepsilon > 0$ (be a "bad" epsilon) be such that $\forall \delta > 0\ \exists x \in A$ s.t. $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$. For $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$, and let $a_n \in A$ be such that $|a_n - a| < \delta_n = \frac{1}{n}$ and $|f(a_n) - f(a)| \geq \varepsilon$. But then $\{a_n\}_{n \geq 1} \subseteq A$, $a_n \to a$, and $f(a_n) \not\to f(a)$. Thus f must be cts at $a \in A$. ▣