Cor. (1.14 Suppose $\{s_n\}_{n\geq 1} \uparrow$, $\{v_n\}_{n\geq 1}$ any sequence s.t. $\exists N$ s.t. $\forall n \geq N$, $s_n \leq v_n$.

Then 1) $\{v_n\}_{n\geq 1}$ converges $\Rightarrow$ $\{s_n\}_{n\geq 1}$ conv. (Since $v_n \to v$ $\Rightarrow \{v_n\}$ bdd $\Rightarrow \{s_n\}$ bdd ...)

2) $\{s_n\}_{n\geq 1}$ diverges $\Rightarrow$ $\{v_n\}$ diverges. (pf ex.)

Defn (Infimum) If $B \subset \mathbb{R}$ is . . . , an infimum of B (g.l.b.) is a real number η s.t.

1) $\eta \leq x$, $\forall x \in B$

2) $\forall \varepsilon > 0$, $\exists x \in B$ s.t. $\eta \leq x < \eta + \varepsilon$.

Thm . . , If B is bdd below, $B \neq \emptyset$, inf B exists and is unique

Pf. Let $A := -B = \{-x \mid x \in B\}$. Then A is bdd above, $A \neq \emptyset$ so $s = \sup A$ exists. Let $\eta = -s$. Then $\eta = \inf B$ (ex.). Uniqueness of inf B follows from uniqueness of sup A. $\boxtimes$.

Cor $\{t_n\}_{n\geq 1} \downarrow$ (decreasing), bdd below $\Rightarrow$ $\{t_n\}_{n\geq 1}$ converges,

$$\lim_{n\to\infty} t_n = \inf \{t_n\}_{n\geq 1}.$$

e.g. (OGO this a.m.) $s_n = q^n$. $A = \{q^n \mid n \in \mathbb{N}\}$.

$q > 1$: $\inf A = 1$, $\sup A$ doesn't exist

$q = 1$: $\inf A = \sup A = 1$

$0 < q < 1$: $\inf A = 0$, $\sup A = 1$

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e.g. Let $x > 0$; $s_n = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n!}$

Since $x > 0$, $s_{n+1} > s_n$ $\forall n$, so $\{s_n\}_{n \geq 1} \uparrow$.

<u>Claim</u> $\{s_n\}_{n\geq 1}$ is bdd above.

Pick $M \in \mathbb{N}$ s.t. $2x \leq M$. Then $\frac{x}{M} \leq \frac{1}{2}$. Set $P = S_{M-1}$.

Let $L = \frac{x^M}{M!}$, $n = M + k$, $k \in \mathbb{N}$. $\frac{x^n}{n!} = x^1$

Then $\frac{x^n}{n!} = \frac{\overbrace{x \cdot x \cdot \quad \cdot x}^{k}}{(M+k)(M+k-1)\dots(M+1)} \frac{x^M}{M!} < \frac{1}{2^k} \cdot L$.

Hence for $n \geq M$, $s_n = P + \frac{x^M}{M!} + \dots + \frac{x^n}{n!} \leq P + L\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-M}}\right)$

$n = M + k$
$k \in \mathbb{N}$

$= P + L \cdot \frac{1 - \left(\frac{1}{2}\right)^{n-M+1}}{1 - \frac{1}{2}}$

$\leq P + 2L$!

Thus, $\lim_{n\to\infty} s_n$ exists. <u>Def^n</u> $e^x = \lim_{n\to\infty} s_n(x)$.

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e.g. Suppose $p > 1$; Consider $\{p^{1/n}\}_{n \geq 1}$. (Def^n of "$p^{1/n}$" late...

$p^{1/n} > 1$, and $p^{1/n} \downarrow$ (ex.). Write $p^{1/n} = 1 + h_n$ with $h_n > 0$.

Then $p = (1+h_n)^n \geq 1 + nh_n \geq nh_n$ $\therefore$ $h_n \leq \frac{p}{n}$

$1 < p^{1/n} = 1 + h_n \leq 1 + \frac{p}{n}$.

This implies (ex.) that $\lim_{n\to\infty} p^{1/n} = 1. = \inf\{p^{1/n}\}_{n\geq 1}$.

(<u>Ex.</u> $0 < p < 1 \Rightarrow \lim_{n\to\infty} p^{1/n} = 1$.)

e.g. $\{n^{1/n}\}_{n \geq 1}$. (OHD next week?) is decreasing, bdd below

$\lim n^{1/n} = 1$.

Accumulation points & Subsequences (P. 183)

Bdd sequences don't always converge, but there is always a way to "pick out a subsequence" that will.

<u>Def^n</u> Let $\{s_n\}_{n\geq 1}$ be a sequence. Another sequence $\{t_k\}_{k\geq 1}$ is a <u>Subsequence</u> of $\{s_n\}_{n\geq 1}$ if

there is a function $\sigma : \mathbb{N} \to \mathbb{N}$ s.t.

1) $n < m \implies \sigma(n) < \sigma(m)$ (ie σ is an "increasing" f^n, "order preserving")

2) $t_k = s_{\sigma(k)}$.

We usually write $\sigma(k) = n_k$ and then denote t_k by s_{n_k}. ie. $\{t_k\}_{k\geq 1} = \{s_{n_k}\}_{k\geq 1}$.

e.g. If $s_n = \frac{1}{n}$, define $s_{n_k} = \frac{1}{2k}$ (i.e., $\sigma(k) = 2k$)

$\{\frac{1}{2k}\}_{k\geq 1}$ is a subsequence.

$a_n = \frac{n}{n+1}$; $a_{n_k} = \frac{k!}{k!+1}$ $(\sigma(k) = k!)$

$b_{n_k} = \frac{k+1}{k+2}$ $(\sigma(k) = k+1)$.

Thm 1.17 Bolzano–Weierstrass (1874) A bdd sequence has at least one convergent subsequence.

L7 L8 09 **Pf.** Suppose $\{s_n\}$ is bdd by L. We subdivide $[-L, L]$ by bisection, creating a sequence of nested interval choosing an element of the sequence in each interval.

x_1 $[$ x_2 $[$ y_1 $]y_2$ $|s_n| \le L$

$[-L, 0, L]$, $\frac{L}{2}$

Now an infinite number of terms of sequence are in $[0, L]$ or $[-L, 0]$.
Choose one of these intervals, call it $[x_1, y_1]$, and choose $n_1 \in \mathbb{N}$ s.t. $s_{n_1} \in [x_1, y_1]$. Bisect $[x_1, y_1]$.
At least one of the 2 subintervals (called $[x_2, y_2]$) contains infinitely many terms of sequence. Choose $n_2 > n_1$ s.t. $s_{n_2} \in [x_2, y_2]$. Note: $|x_2 - y_2| = \frac{2L}{2^2}$, $[x_2, y_2] \subset [x_1, y_1]$. ($\{j \mid s_j \in [x_i, y_i]\}$ is infinite ..., $\forall i \ge 1$).
Continue in this way (inductively, if you wish), choosing a subsequence $\{s_{n_k}\}_{k \ge 1}$ (i.e. s.t. $n_{k+1} > n_k$, $\forall k \in \mathbb{N}$) with $s_{n_k} \in [x_k, y_k] \subset \bigcap_{j \le k} [x_{n_j}, y_{n_j}]$ and $|x_k - y_k| = \frac{2L}{2^k}$. Hence, $x_k \le s_{n_k} \le y_k$. By the nested interval theorem, as $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = s \in \bigcap_{j \in \mathbb{N}} [x_j, y_j]$.
Then, $\lim_{k\to\infty} s_{n_k} = s$. (Let $\varepsilon > 0$; Choose N s.t. $|x_k - s| < \varepsilon$, $|y_k - s| < \varepsilon$ $\forall k \ge N$.
$\Rightarrow [x_k, y_k] \subset (s-\varepsilon, s+\varepsilon)$ $\therefore$ $s_{n_k} \in (s-\varepsilon, s+\varepsilon) \Leftrightarrow |s - s_{n_k}| < \varepsilon$, $\forall k \ge N$ ∎

Def^n A real number s is an <u>accumulation point</u> of a sequence $\{s_n\}_{n \ge 1}$ if there is a subsequence $\{s_{n_k}\}_{k \ge 1}$ converging to s, i.e. $\lim_{k\to\infty} s_{n_k} = s$.

Thus, B-W may be restated as "<u>Every bdd sequence has at least one accumulation pt.</u>"

See the book for another pf: idea: Let $A = \{x \in [-K, K] \mid$ there are infinitely many terms of $\{s_n\}_{n \ge 1}$ in $[x, K]\}$. Then $A \ne \emptyset$ ($-K \in A$), A is bdd above by K, so $t = \sup A$ exists. (ex, or read book): t is an accumulation pt of $\{s_n\}_{n \ge 1}$ (in fact, the "largest"; ("$\limsup s_n$, $\sup s_n$"; later
e.g. $\{(-1)^n\}$ has 2 accumulation pts, 1 & -1. N.B. Having just 1 acc. pt $\not\Rightarrow$ convergence. e.g. $a_n = \begin{cases} 1 + \frac{1}{n}, & n \text{ even} \\ n & \text{odd} \end{cases}$
See book P. 185 for an example / seen before last week) of a sequence in $[0, 1]$ for which <u>every</u> real number in $[0, 1]$ is an accumulation pt. "$\{s_n\}_{n \ge 1}$ is dense in $[0, 1]$". $0, \frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

Exercise: If $s_n \to s$, then every subsequence of $\{s_n\}_{n \ge 1}$ also converges to s!
(indeed, trivially, "iff").

III.2 Series

Suppose $\{a_n\}_{n\geq 1}$ is a sequence (in $\mathbb{R}$).

Let $S_n := \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$.

Def^n The series "$\sum_{k=1}^{\infty} a_k$" converges if $\{S_n\}_{n\geq 1}$ converges.

If $S_n \to S$, we write $\sum_{k=1}^{\infty} a_k = S$. (If $\{S_n\}_{n\geq 1}$ does not converge we say $\sum_{k=1}^{\infty} a_m$ ($\sum a_m$) does not.) We will also write $\sum_{n\geq 1} a_n$ converges. $(a_n \in \mathbb{N})$

e.g. $a_n = \frac{1}{2^{n-1}}$ $\quad S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$. We know $S_n = \dfrac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}$

$= 2 \cdot - \frac{1}{2^{n+1}}$

Hence, $\lim_{n\to\infty} S_n = 2$ so so $\sum_{n\geq 1} \frac{1}{2^{n-1}} = 2$ or $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.

e.g Indeed if $a_n = A.q^n$ and $0 < q < 1$, then $\sum_{n\geq 1} a_n$ converges to $A.\left(\frac{q}{1-q}\right)$.

e.g. $a_n = \frac{1}{n(n+1)}$, $\quad S_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$

Recall: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ $\therefore S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1} \to 1.$

Hence $\sum_{n\geq 1} \frac{1}{n(n+1)} = 1.$

e.g. (Dan #3 19-9-~~08~~ -09 #4) $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ "Harmonic series"

$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

$S_{2^n} = \underbrace{1 + \frac{1}{2}}_{>\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{>\frac{1}{2}} + \cdots + \underbrace{\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n}}_{>\frac{1}{2}}$

$> \frac{n}{2}$. Hence $\sum_{n\geq 1} \frac{1}{n}$ diverges.

L8 Remark $\{s_n\}$ seq. of partial sums of $\sum a_n$; $\{S_n\}$ conv $\Leftrightarrow$ $\{S_n\}$ Cauchy. 25.

Prop^n 2.3 $\sum a_n$ conv. $\Leftrightarrow$ $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N, \forall p \geq 1$ $(m = p + \ldots$

$(|S_{n+p} - S_n| =)$ $|a_{n+1} + \cdots + a_{n+p}| < \varepsilon$

Pf. Clear

Cor. If $\sum_{n \geq 1} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. (Put $p=1$ in Prop 2.3).

(Hence, cd replace criterion by "... $|a_n + \cdots + a_{n+p}| < \varepsilon$"!

Remark The converse of the Cor. is not true i.e. $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum a_n$ conv.,

Since (e.g.) $\frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges.

Series with positive terms (Why? $S_n \uparrow$!)

Suppose $\{a_n\}_{n \geq 1} \subset [0, \infty)$ i.e. $a_n \geq 0, \forall n \in \mathbb{N}$. Then $S_n = \sum_{j=1}^{n} a_j$.

satisfies $S_{n+1} = S_n + a_n \geq S_n$, i.e. $\{S_n\}_{n \geq 1}$ is increasing.

We know that an increasing sequence converges $\Leftrightarrow$ it is bdd above!

i.e. $\sum a_n$ converges $\Leftrightarrow$ its partial sums are bounded above!
$(a_n \geq 0)$

N.B. Comparison Theorem 2.5 If $\exists N$ s.t. $0 \leq a_n \leq b_n$ for $n \geq N$, then

1) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges
2) $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges.

Pf 1) $\sum b_n$ converges $\Rightarrow$ partial sums of $\sum b_n$ conv $\Rightarrow$ partial sums of $\sum b_n$ are bdd. But $a_n \leq b_n$ $\Rightarrow$ partial sums of $\sum a_n$ are bdd by partial sums of $\sum b_n$, hence partial sums of $\sum a_n$ are bdd above.

2) (Contrapositive of 1).)

e.g. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$; $0 \leq a_n = \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$; we know $\sum \frac{1}{n(n+1)}$ conv.

$\therefore \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ conv. i.e. $\frac{1}{2^2} + \frac{1}{3^2} + \cdots$ conv.

$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ conv i.e. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\therefore$ if $k \geq 2$ $\sum \frac{1}{n^k}$ conv ...

Ratio Test (Cauchy : cf 2.10) $\{a_n\}_{n\geq 1}$ $\begin{pmatrix} \text{s.t.} & a_n \geq 0 & \forall n \\ \exists N \text{ s.t.} & a_n > 0 & , n \geq N. \end{pmatrix}$

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$, and

1) $q < 1$, then $\sum a_n$ conv,
2) $q > 1$, then $\sum a_n$ div.,
3) $q = 1$, anything can happen.

Pf. (outline) 1) $q < 1 \Rightarrow \exists N$ s.t. $\frac{a_{n+1}}{a_n} < r < 1$ for some r, $\forall n \geq N$. $\quad q \; r \; 1$ $\quad$ (e.g. $\varepsilon = \frac{1-q}{2} > 0$)

Now compare with series $\left(\frac{a_N}{r^N}\right) \cdot r^n$, which is conv. ...

$(a_{N+k} < r^k a_N$ i.e. $a_n < r^{n-N} a_N = \frac{r^n}{r^N} a_N$ for $n \geq N+1$. ..)

2) $q > 1 \Rightarrow \exists N$ s.t. $\frac{a_{n+1}}{a_n} > r > 1$ for all $n \geq N$. Then $\underset{n\geq N+1}{a_n} > a_N$, so $a_n \not\to 0$. Hence $\sum a_n$ diverges.

3) • $\sum \frac{1}{n(n+1)}$ converges, and $\frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+1)(n+2)} = \frac{n}{n+2} \to 1$.

• $\sum \frac{1}{n}$ diverges, and $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$.

Root Test (Cauchy) (a_n : "eventually positive").

If $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ and
1) $r < 1$, then $\sum a_n$ conv.
2) $r > 1$, then $\sum a_n$ diverges
3) $r = 1$, anything can happen.

Pf. (Outline) 1) $r < 1 \Rightarrow \exists N$ s.t. $\forall n \geq N$ $0 \leq \sqrt[n]{a_n} \leq s < 1$ (for some $s < 1$.)
$\Rightarrow 0 \leq a_n \leq s^n$, $s < 1$. Now use comparison.

2) $r > 1 \Rightarrow \exists N$ s.t. $\forall n \geq N$, $\sqrt[n]{a_n} \geq 1 \Rightarrow a_n \geq 1$, $\forall n \geq N$ ...

3) • $\sum \frac{1}{n^2}$ conv, and $\sqrt[n]{\frac{1}{n^2}} = \frac{1}{(n^{\frac{1}{n}})^2} \to 1$

• $\sum \frac{1}{n}$ diverges, and $\sqrt[n]{\frac{1}{n}} \to 1$ too. $\quad$ [$n^{\frac{1}{n}} = 1 + h_n$, $h_n > 0$, so $n = (1+h_n)^n \geq \frac{n(n-1)}{2} h_n^2$ ∴ ...

e.g. $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ — $\begin{cases} s \leq 1 \Rightarrow \text{series diverges} \\ s > 1 \Rightarrow \text{series converges} \end{cases}$

($S=1$ - seen; $s<1 \Rightarrow \frac{1}{n^s} > \frac{1}{n}$, so by comparison, the series $\sum \frac{1}{n^s}$ diverges)

Let $p_n :=$ nth partial sum $= \sum_j^n \frac{1}{j^s}$

Let $k \in \mathbb{N}$; the $P_{2^{k+1}} = 1 + \underbrace{\frac{1}{2^s} + \frac{1}{3^s}} + \underbrace{\frac{1}{4^s} + \cdots + \frac{1}{8^s}} + \cdots + \frac{1}{(2^k)^s} + \cdots + \frac{1}{(2^{k+1})^s}$

$$< 1 + \frac{2}{2^s} + \frac{4}{4^s} + \frac{8}{8^s} + \cdots + \frac{2^k}{(2^k)^s}$$

$$= 1 + \frac{1}{2^{s-1}} + \underbrace{\frac{1}{4^{s-1}}}_{\left(\frac{1}{2^{s-1}}\right)^2} + \left(\frac{1}{2^{s-1}}\right)^3 + \left(\frac{1}{2^{s-1}}\right)^k.$$

$\therefore P_{2^{k+1}} < \sum_{j=0}^{k} \left(\frac{1}{2^{s-1}}\right)^j$

But if $s>1$, $2^{s-1} > 1$ so $\frac{1}{2^{s-1}} < 1$. Hence $\sum_{j=0}^{\infty} \left(\frac{1}{2^{s-1}}\right)^j$ converges and hence is bdd. Thus, p_m is bdd (by $\frac{1}{1-(\frac{1}{2})^{s-1}}$!) $\therefore \sum \frac{1}{n^s}$ conv.

Series with arbitrary terms

Def^n $\sum_{n\geq 1} a_n$ is absolutely convergent if $\sum_{n\geq 1} |a_n|$ converges.
(converges absolutely)

Prop^n A series which converges absolutely, converges. (absolutely!)

Pf. We consider 4 series constructed from $\{a_n\}$:

(I) $\sum a_n$

(II) $\sum |a_n|$

(III) $\sum b_n$, where $b_n = \max\{a_n, 0\}$ (so $b_n \geq 0$) (picks out +ve term of $\sum a_n$)

(IV) $\sum c_n$, " $c_n = \max\{-a_n, 0\}$ (so $c_n \geq 0$; picks out negative terms, changes sign.

Then $a_n = b_n - c_n$ and $|a_n| = b_n + c_n$; If $S_n^{I}, S^{II}, S^{III}, S^{IV}$ denote the nth partial sum of each series. Then note that II, III & IV are series with positive terms. As well

$S_n^{-} = S_n^{III} - S_n^{IV}$, $S_n^{II} = S_n^{III} - S_n^{IV}$

By hypothesis, S_n^{II} converges. Hence $\{S_n^{II}\}$ is bdd. Hence, both $\{S_n^{III}\}$ & $\{S_n^{IV}\}$ are bdd too! Hence $\{S_n^{III}\}$, $\{S_n^{IV}\}$ both converge. Hence $S_n^{III} \to S^{III}$, $S_n^{IV} \to S^{IV}$, so that

$$(\Sigma a_n): \quad S_n^{I} \to S^{III} - S^{IV}. \quad \therefore \Sigma a_n \text{ converges}$$

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N.B. <u>Corollary of pf.</u> 1) If a series converges absolutely, both the series of positive terms <u>and</u> the series of negative terms converge, and the series converges to their sum.

2) If a series converges, but <u>not</u> absolutely, both the series of positive terms <u>and</u> the series of negative terms diverge.

$S_n = S_n^{+} + \underbrace{S_n^{-}}_{-S_n^{IV}}$; $S_n^{+} = \frac{1}{2}(S_n + S_n^{abs})$

$S_n^{abs} = S_n^{+} \underbrace{- S_n^{-}}_{+S_n^{IV}}$; $S_n^{-} = \frac{1}{2}(S_n - S_n^{abs})$.

Σa_n	1	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$	$+\frac{1}{5}$	$-\frac{1}{6}$ ...
$+$	1	$+0$	$+\frac{1}{3}$	$+0$	$+\frac{1}{5}$	$+0$
$-$	0	$-\frac{1}{2}$	$+0$	$-\frac{1}{4}$	$+0$	$-\frac{1}{6}$
abs	1	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{4}$	$+\frac{1}{5}$	$+\frac{1}{6}$

<u>Theorem</u> (Leibniz 1682) Suppose $\{a_n\}_{n\geq 1} \downarrow$, $a_n \geq 0$ & $\lim_{n\to\infty} a_n = 0$.

Then, $\sum_{n\geq 0} (-1)^n a_n \longrightarrow S$ (converges), and $|S - S_n| \leq a_{n+1}$, $n \geq 0$.

S_n ↑ nth partial sum

<u>Pf.</u> Let $S_n := \sum_{k=0}^{n} (-1)^k a_k = a_0 \overbrace{- a_1 + a_2}^{\leq 0} \overbrace{- a_3 + a_4}^{\leq 0} - a_5$

$= \underbrace{a_0 - a_1}_{\geq 0} + \underbrace{a_2 - a_3}_{\geq 0} + \underbrace{a_4 - a_5}_{\geq 0}$

← $S_{2n} \downarrow \leq a_0$

$S_{2n+1} \leq S_{2n}$

← $S_{2n+1} \uparrow$

Then,
$$S_{2n+2} = S_{2n} - a_{2n+1} + a_{2n+2} \leq S_{2n}$$
$$= S_{2n+1} + a_{2n+2} \geq S_{2n+1}$$
$$S_{2n+1} = S_{2n-1} + a_{2n} - a_{2n-1} \geq S_{2n-1}$$

i.e.
$$S_{2n-1} \leq S_{2n+1} \leq S_{2n+2} \leq S_{2n}.$$