Hence $\alpha \neq \tilde{0}$ means if $\alpha = [\{a_n\}_{n\geq 1}]$,
$\exists N \in \mathbb{N}$, $\exists r > 0$ s.t. $\forall p \geq N$, $|\alpha| > r$.

Define $\alpha^{-1} = \frac{1}{\alpha}$ as follows: Let $b_p = \begin{cases} 0, & p \leq N-1 \\ \frac{1}{a_p}, & p \geq N \end{cases}$

Then (exercise). $\alpha^{-1} := [\{b_p\}_{p\geq 1}]$ satisfies $\alpha^{-1} \cdot \alpha = \tilde{1}$.

(Remark; $b_p . a_p = \begin{cases} 0, & p \leq N-1 \\ 1, & p \geq N \end{cases}$! $\{b_p\}\{a_p\} = (0, 0, \dots, \overset{\downarrow N^{th}\text{ pos}}{1}, 1, 1 \dots)$ 1's forever

and $(0, 0, \dots, \overset{N^{th}}{1}, 1, \dots) \sim (1, 1, \dots).$)

Field axioms are now straightforward exercises.

Next step: Order rel^n. "$\alpha \leq \beta$"? α, β real numbers.

We know $\alpha \neq \tilde{0} \Leftrightarrow$ $\exists r > 0$, $\exists N \in \mathbb{N}$ either I) $\forall p \geq N$, $a_p > r$
II) $\forall p \geq N$, $a_p < -r$.

Def^n 1) $\mathbb{R}^+ = \{\alpha \in \mathbb{R} \mid$ (I) holds for α, or, $\alpha = \tilde{0}\}$.

2) If $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta \Leftrightarrow \beta - \alpha \in \mathbb{R}^+$

L4 3) If $\alpha \in \mathbb{R}$, $|\alpha| := \max\{\alpha, -\alpha\}$.

Remark ① (Exercise) $|\alpha| < \varepsilon \Leftrightarrow \exists r > 0, \exists N$ s.t. $n \geq N \Rightarrow |a_n| + r < \varepsilon$ ② $|[\{a_n\}_{n\geq 1}]| = [\{|a_n|\}_{n\geq 1}]$
(or" $n \geq N \Rightarrow |a_n| < \varepsilon - r$
$\alpha = [\{a_n\}_{n\geq 1}]$ ($\varepsilon \in \mathbb{Q}$ for now)

Exercise. The triangle inequality holds; all usual properties of "$\leq$" hold.

Theorem A. If $\alpha = [\{a_n\}_{n\geq 1}]$, then, in $\mathbb{R}$, $\lim_{n\to\infty} \tilde{a}_n = \alpha$.

Pf. $\alpha = [(a_1, a_2, \dots)]$ $\tilde{a}_n = [(a_n, a_n, \dots)]$
$\alpha - \tilde{a}_n = [(a_1 - a_n, a_2 - a_n, \dots)]$, so $|\alpha - \tilde{a}_n| = [(|a_1 - a_n|, |a_2 - a_n|, \dots, |a_p - a_n|, \dots$

Let $\varepsilon > 0$; Since $\{a_n\}_{n\geq 1}$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. $\forall n, p \geq N$, $|a_p - a_n| < \frac{\varepsilon}{2}$.

Then, $\forall n \geq N$, $|\alpha - \tilde{a}_n| < \varepsilon$, since $\alpha - \tilde{a}_n = [\{|a_p - a_n|\}_{p\geq 1}]$ and $\forall p \geq N$ $|a_p - a_n| < \varepsilon - \frac{\varepsilon}{2}$ ☒

Hence All Cauchy sequences of rational numbers
(⇒) converge in $\mathbb{R}$!

Rmk: if $\{a_n\}_{n\geq 1}$ is Cauchy, $\Leftrightarrow$ $\{\tilde{a}_n\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$
in $\mathbb{Q}$

$(n,p\geq N \Rightarrow |a_n-a_p|<\varepsilon)$ $\Rightarrow$ $(n,p\geq N \Rightarrow |\tilde{a}_n-\tilde{a}_p| = [(a_n-a_p, a_n-a_p),\dots] < \varepsilon$ $N\neq 1$!

Big Theorem Every Cauchy sequence of real numbers converges to a real number! ("$\mathbb{R}$ is complete")
(1.8, P175

Proof: We'll do this in 3 steps. The first will use a very useful lemma:

Step 1: Approximation of reals by rationals

Lemma A If $\alpha\in\mathbb{R}$, $\forall\varepsilon>0$, $\exists q\in\tilde{\mathbb{Q}}$ s.t. $|\alpha-q|<\varepsilon$

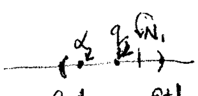
pf. Suppose $\alpha = [\{a_n\}_{n\geq 1}]$. Since $\{a_n\}_{n\geq 1}$ is Cauchy $\exists N$ s.t. $\forall p,n\geq N$, $|a_n-a_p|<\frac{\varepsilon}{2}$. In particular, $|a_p-a_N|<\frac{\varepsilon}{2}$, $\forall p\geq N$. Let $q = \tilde{a}_N = (a_N, a_N, a_N, \dots)$

Then $|\alpha-q| = (|a_1-a_N|, |a_2-a_N|, \dots, |a_p-a_N|, \dots)$

But $\forall p\geq N$, $|a_p-a_N|<\frac{\varepsilon}{2}$, so, $\exists N (=N!)$ s.t. $\forall p\geq N$, $|a_p-a_N|<\varepsilon-\frac{\varepsilon}{2}$.

$\therefore |\alpha-q|<\varepsilon$. ▨ lemma A.

Corollary (Archimedian property for $\mathbb{R}$). $\forall\alpha\in\mathbb{R}$, $\exists N\in\tilde{\mathbb{N}}$ s.t. $N>\alpha$.



Pf. Let $\varepsilon=1$ and choose $q\in\tilde{\mathbb{Q}}$ s.t. $|\alpha-q|<1$. Now choose $N_1\in\mathbb{N}$ s.t. $N_1\geq q$. (Done in a DwD). Then, set $N = [(N_1+1, N_1+1, \dots] = \widetilde{N_1+1}$.
Thus, $\alpha\in(q- \ , q+ \ )$, so $\alpha<q+1\leq\tilde{N}_1+\tilde{1} = N$. ▨

Now let $\{\alpha_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers

Step 2:

Lemma B There is a Cauchy sequence of rational numbers $\{q_n\}_{n\geq 1}$ s.t. $|\alpha_n-q_n|<\frac{1}{n}$, $\forall n\in\mathbb{N}$.

Let $n \in \mathbb{N}$.

Pf. For the real number α_n, by lemma A, with $\varepsilon = \frac{1}{n}$, $\exists q_n \in \mathbb{Q}$ s.t. $|\alpha_n - q_n| < \frac{1}{n}$. (16)

We now show that $\{q_n\}_{n\geq 1}$ is Cauchy:

Let $\varepsilon > 0$. Since $\{\alpha_n\}_{n\geq 1}$ is Cauchy, $\exists N_0 \in \mathbb{N}$ s.t. $n,p \geq N_0 \Rightarrow |\alpha_n - \alpha_p| < \frac{\varepsilon}{3}$. Now choose N_1 s.t. $\frac{1}{N_1} < \frac{\varepsilon}{3}$ (ie $N_1 > \frac{3}{\varepsilon}$), and let $N := \max(N_0, N_1)$. Then, $\forall n,p \geq N$, $\frac{1}{n}, \frac{1}{p} < \frac{1}{N} \leq \frac{1}{N_1} < \frac{\varepsilon}{3}$, and $n,p \geq N_0$, so $|\alpha_n - \alpha_p| < \frac{\varepsilon}{3}$.

Hence, $n,p \geq N \Rightarrow$
$$\begin{aligned}|q_n - q_p| &= |q_n - \alpha_n + \alpha_n - \alpha_p + \alpha_p - q_p| \\ &\leq |q_n - \alpha_n| + |\alpha_n - \alpha_p| + |\alpha_p - q_p| \\ &< \frac{1}{n} + \frac{\varepsilon}{3} + \frac{1}{p} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus, $\{q_n\}_{n\geq 1}$ is Cauchy. ☒ lemma B

Now define $\alpha = [\{q_n\}_{n\geq 1}]$, where $\{q_n\}_{\geq 1}$ has been chosen for $\{\alpha_n\}_{n\geq 1}$ by lemma B. (So $\{q_n\}_{n\geq 1}$ is Cauchy, & $|\alpha_n - q_n| < \frac{1}{n}$, $\forall n \geq 1$.)

Step 3: We show $\lim_{n\to\infty} \alpha_n = \alpha$: We know $\lim_{n\to\infty} q_n = \alpha$ by Theorem A.

So, this is now easy! Let $\varepsilon > 0$, and choose N_0 s.t. $n \geq N_0 \Rightarrow |\alpha - q_n| < \frac{\varepsilon}{2}$. Now, choose N_1 s.t. $\frac{1}{N_1} < \frac{\varepsilon}{2}$; and let $N = \max(N_0, N_1)$.

Then, $n \geq N \Rightarrow$
$$\begin{aligned}|\alpha - \alpha_n| &= |\alpha - q_n + q_n - \alpha_n| \\ &\leq |\alpha - q_n| + |q_n - \alpha_n| \\ &< \frac{\varepsilon}{2} + \frac{1}{n} \\ &\leq \frac{\varepsilon}{2} + \frac{1}{N_1} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$
☒

Hence, it all stops here.

Remark: Every convergent sequence of real numbers is Cauchy.
(Pf: word for word the same as in $\mathbb{Q}$.)

Thus, $\{a_n\}_{n\geq 1} \in \mathbb{R}$ is convergent $\iff$ it is Cauchy!

(does not hold for $\mathbb{Q}$!)

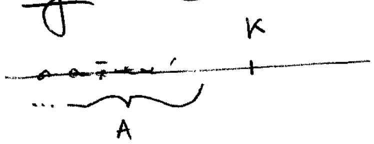
Supremum, Infimum. ("Sup", "inf") ("lub", "glb")

A property that is equivalent to the completeness of $\mathbb{R}$ is the existence of "least upper bounds" for sets of reals which are "bdd. above"

Def^n ① $A \subseteq \mathbb{R}$ is bounded above (below) if $\exists K \in \mathbb{R}$ s.t.

$\forall x \in A, \quad x \leq K.$ (K is called an upper bound for A)

($\forall x \in A, \quad K \leq x.$) (K is called a lower bound for A)



② $A \subseteq \mathbb{R}$ is bounded if it is bdd above and below

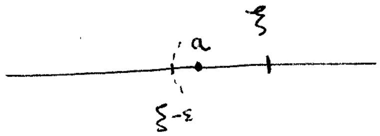
(ex. $\iff \exists L > 0$ s.t. $\forall x \in A \quad |x| \leq L$.).

N.B. ③ Let A be a subset of $\mathbb{R}$. A real number $\xi \in \mathbb{R}$ is a least upper bound or a supremum of A if

1) $\forall x \in A, \quad x \leq \xi$ (is an upper bound)

and

2) $\forall \varepsilon > 0, \exists\, x \in A$ with $\xi - \varepsilon < x$. (is the least upper bound)



We write $\xi = \sup A$

ex. 2. Suppose $\sup A$ exists. Prove that there is a sequence $\{a_n\}_{n\geq 1} \subset A$ s.t. $a_n \leq a_{n+1}$ (increasing), $\lim_{n\to\infty} a_n = \sup A$.

ex. 0 Show that if $\sup A$ exists, it is unique!

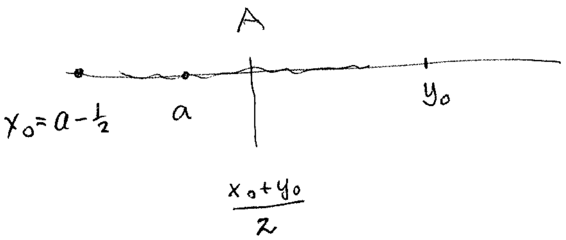
ex 1 Suppose $A \subset \mathbb{R}$, σ satisfies 1) σ is an u.b. for A
2) $\forall b$ s.t. b is an u.b. for A, $\sigma \leq b$.
Then $\sigma = \sup A$.

e.g. $A = \{x \in \mathbb{R} \mid x^2 < 2\}$; $\sup A = \sqrt{2}$!

e.g. There are some sets which cannot have a supremum, since they are not bdd above e.g. $A = \{n \mid n \in \mathbb{N}\}$ cannot have a sup.

Thm (1.21) P.181 If $\emptyset \neq A \subset \mathbb{R}$ is bdd above, $\sup A$ exists.

Pf. "By bisection". Let y_0 be an upper bound for A. Choose $a \in A$ and let $x_0 = a - 1$.



Then

1) x_0 is not an upper bound for A
2) y_0 is an upper bd. for A.
3) $x_0 < y_0$

Now define $x_1 = \begin{cases} \frac{x_0+y_0}{2} & \text{if } \frac{x_0+y_0}{2} \text{ is not an u.b. for A} \\ x_0 & \text{if } \text{'' is an ''.} \end{cases}$

$y_1 = \begin{cases} y_0 & \text{if } \frac{x_0+y_0}{2} \text{ is not an u.b. for A} \\ \frac{x_0+y_0}{2} & \text{if } \text{'' is an u.b. for A.} \end{cases}$

Then 1) x_1 is not an u.b. for A

2) y_1 is an u.b. for A

3) $|x_1 - y_1| = \frac{1}{2}|x_0 - y_0|$

4) $x_1, y_1 \in [x_0, y_0]$. i.e. $[x_1, y_1] \subset [x_0, y_0]$.

Continue recursively in this way, we can construct 2 sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n \geq 1}$ s.t.

1) x_n is not an u.b. for A

2) y_n is an u.b. for A

3) $|x_n - y_n| = \frac{1}{2}|x_{n-1} - y_{n-1}| = \frac{1}{4}|x_{n-2} - y_{n-2}| = \cdots = \frac{1}{2^n}|x_0 - y_0|$

4) $[x_n, y_n] \subset [x_{n-1}, y_{n-1}] \subset [x_{n-2}, y_{n-2}] \subset \cdots$

i.e. $[x_n, y_n] \subset \bigcap_{0 \leq j \leq n} [x_j, y_j]$

We now continue by showing that $\lim_{n\to\infty} x_n = \lim_{n \to \infty} y_n =: \xi$ exists.

First, let $n, k \in \mathbb{N}$. Then $|x_n - x_{n+k}| \leq |x_n - y_n| = \frac{|x_0 - y_0|}{2^n}$ &

$|y_n - y_{n+k}| \leq \frac{|x_0 - y_0|}{2^n}$.

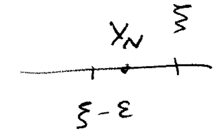
Hence, both sequences $\{x_n\}_{n\geq 1}$ & $\{y_n\}_{n \geq 1}$ are Cauchy! (Let $\varepsilon > 0$ and (ex.) choose N s.t. $\frac{|x_0 - y_0|}{2^N} < \varepsilon$. Then, $\forall\, n, p \geq N$, $|x_n - x_p| < \varepsilon$.).

($p = n+k$, say...)
($n \geq N$ ...)

Hence, $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist. Because of (3), they are the same, say ξ.

We conclude by proving that $\xi = \sup A$.

① ξ is an u.b. for A: Let $a \in A$. Then by (3), $a \le y_n$, $\forall n$. By a result from a DGR, this implies $a \le \lim_{n\to\infty} y_n = \xi$. Hence, $\forall a \in A$, $a \le \xi$. Thus ξ is an u.b. for A.



② ξ is the least upper bound. Let $\varepsilon > 0$.

Since $x_n \to \xi$, for $\varepsilon' = \varepsilon$, $\exists N$ s.t. $x_N \in (\xi-\varepsilon, \xi+\varepsilon)$. But by (1), $\forall N$, X_N is not an u.b. for A, so $\exists a \in A$ s.t. $X_N < a$, and so $\xi - \varepsilon < a$.

for $\varepsilon > 0$, $\exists a \in A$ s.t. $\xi - \varepsilon < a \le \xi$. ☒

Exercise "Nested Intervals Theorem"

Suppose $\{[a_n, b_n] \mid n \ge 1\}$ is a sequence of nested closed intervals

i.e. $[a_n, b_n] \subset \bigcap_{j \le n} [a_j, b_j]$ ("nested")

and $\lim_{n\to\infty} (a_n - b_n) = 0$. Then, 1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =: L$ exist

2) $\{L\} = \bigcap_{n\in\mathbb{N}} [a_n, b_n]$.

(DGD?) Recall the exercise: Suppose $\sup A$ exists, then $\exists$ increasing sequence $\{a_n\}_{n\ge1} \in A$ s.t. $a_n \to \sup A$.

Thm 1.13 Suppose $\{s_n\}_{n\ge1}$ is an (monotonically) increasing sequence, which is bounded above. Then $\lim_{n\to\infty} s_n = \sup\{s_n \mid n \ge 1\}$ exists.

Pf. Set $A = \{s_1, s_2, \dots\} = \{s_n \mid n \ge 1\}$. By hypothesis, $s := \sup A$ exists.

Let $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ s.t. $\xi - \varepsilon < s_N \le \xi$. But, $\forall n \ge N$, $s_n \ge s_N$

and so $\forall n \ge N$, $s - \varepsilon < s_n \le s$. Hence, $n \ge N \Rightarrow |s - s_n| < \varepsilon$.

Thus $\lim_{n\to\infty} s_n = s = \sup\{s_n \mid n \ge 1\}$ ☒