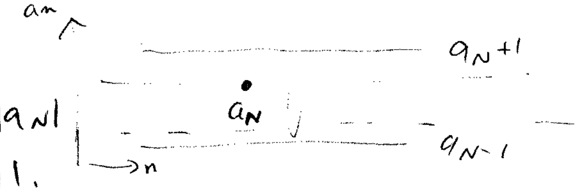
Thm Every Cauchy sequence is bounded.

Pf. (ex?) Let $\{a_n\}_{n\geq 1}$ be Cauchy. Let $\varepsilon = 1$. Then $\exists N$ s.t. $\forall n, p \geq N$, $|a_n - a_p| < 1$.

In particular, $|a_p - a_N| < 1$, $\forall p \geq N$.

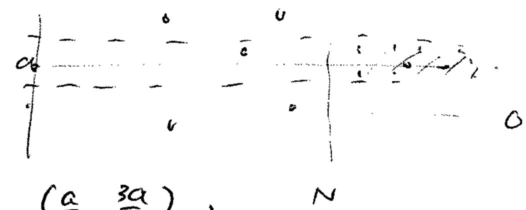


Hence $p \geq N \Rightarrow |a_p| = |a_p - a_N + a_N| \leq |a_p - a_N| + |a_N|$
$< 1 + |a_N|$.

Let $B = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N|+1\}$. Then, $\forall n \geq 1$, $|a_n| \leq B$ ☒

Propⁿ If $\lim_{n\to\infty} a_n = a > 0$, $\exists N$ s.t. $\forall n \geq N$, $a_n > \frac{a}{2} (>0)$

Pf. Let $\varepsilon = \frac{a}{2} > 0$. Then $\exists N$ s.t. $\forall n \geq N$
$|a - a_n| < \varepsilon$. Hence, $a_n \in (a-\varepsilon, a+\varepsilon)$
$= (a - \frac{a}{2}, a + \frac{a}{2}) = (\frac{a}{2}, \frac{3a}{2})$,



In particular, $\forall n \geq N$, $a_n > \frac{a}{2}$. ☒

Remark ① If $a_n \to a < 0$, $\exists N$ s.t. $\forall n \geq N$, $a_n < \frac{a}{2} < 0$.

② If $\lim_{n\to 0} a_n = a \neq 0$, either 1) $\exists N$ s.t $\forall n \geq N$, $a_n > \frac{a}{2} > 0$
or 2) " " $a_n < \frac{a}{2} < 0$.

Cauchy version: Ex. Suppose $\{a_n\}_{n\geq 1}$ is Cauchy and $\lim_{n\to\infty} a_n \neq 0$. (In this case, there's no 'a'! The limit might not exist ...) Prove that

I) $\exists r > 0$ s.t. either 1) $\exists M$ s.t. $\forall p \geq M$, $a_p > r > 0$
2) $\exists M$ s.t. $\forall p \geq M$, $a_p < -r < 0$.

AND

II) $\exists r > 0$ s.t. $\exists M > 0$ s.t. ~~$\forall p \geq M$,~~ $|a_p| \geq r$. (Note (ex.) (I) $\Leftrightarrow$ (II) for Cauchy seq.)

Thm (1.5 P.174) Suppose $a_n \to a$, $b_n \to b$. Then

1) $\lim_{n\to\infty} (a_n + b_n) = a + b$ (i.e. $a_n + b_n \to a + b$)
2) $\lim_{n\to\infty} a_n b_n = ab$ (i.e. $a_n b_n \to ab$)

If in addition, $b \neq 0$, then ($\exists N$ s.t. $\forall n \geq N$, $b_n \neq 0$, and)

3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

Pf. 1) $|a+b-(a_n+b_n)| = |a-a_n+b-b_n| \le |a-a_n| + |b-b_n|$ $\dots$

2) As $a_n \to a$, $\exists A$ s.t. $|a_n| \le A$, $\forall n \ge 1$. (Wlog $A>0$).

$\therefore$ $|ab-a_nb_n| = |ab-a_nb+a_nb-a_nb_n| = |b(a-a_n)+a_n(b-b_n)|$
$\le |b||a-a_n| + A.|b-b_n|$ $\dots$ (Let $\varepsilon>0$. If $b=0$ choose N_b s.t. $\forall n \ge N_b$, $|b-b_n| < \frac{\varepsilon}{A}$; If $b \ne 0$, Choose N_a s.t. $\forall n \ge N_a$ $|a-a_n| < \frac{\varepsilon}{2|b|}$; choose N_b s.t. $\forall n \ge N_b$, $|b-b_n| < \frac{\varepsilon}{2A}$)

$\dots$

3) Using 2, it suffices to show $\frac{1}{b_n} \to \frac{1}{b}$ if $b_n \to b$, and $b \ne 0$.
again, we may assume $b>0$. (If $b<0$, replace $\{b_n\}_{n\ge1}$ by Let $a_n=-1$, use 2).

We know by our prop^n that $\exists N_1$ s.t. $n \ge N_1 \Rightarrow b_n > \frac{b}{2} > 0$.
Hence $\frac{1}{b_n} < \frac{2}{b}$. Thus, $n \ge N_1 \Rightarrow \left|\frac{1}{b} - \frac{1}{b_n}\right| = \frac{|b_n-b|}{bb_n} \le \frac{2}{b^2}|b_n-b|$ $\dots$

"Cauchy" versions! (Exercises)

Thm Suppose $\{a_n\}_{n\ge1}$, $\{b_n\}_{n\ge1}$ are Cauchy. Then

1) $\{a_n+b_n\}_{n\ge1}$ is Cauchy 2) $\{a_nb_n\}_{n\ge1}$ is Cauchy

If, in addition, $\lim_{n\to\infty} b_n \ne 0$, 3) $\left\{\frac{a_n}{b_n}\right\}_{n\ge M}$ is Cauchy.

($\Rightarrow \exists r>0$, $\exists M \in \mathbb{N}$ s.t. $p \ge M \Rightarrow |b_p| > r$)

Pf. 1) $|a_n+b_n-(a_p+b_p)| = |a_n-a_p+b_n-b_p| \le |a_n-a_p| + |b_n-b_p|$ $\dots$

2) Since $\{a_n\}_{n\ge1}$ is bdd, say $|a_n| \le A$, $\forall n \ge 1$, imitate pf of (2) in 1.5

3) Use cauchy version of "Prop^n", and imitate pf of (3) in 1.5.

That is, $b_n > b_{N_0} - \frac{r}{2}$ and $b_n < b_{N_0} + \frac{r}{2}$ and $|b_{N_0}| > r$ (rc)
$\forall n \geq N_0$

Case 1 $b_{N_0} > r$. Then, $\forall n \geq N_0$, $b_n > r - \frac{r}{2} = \frac{r}{2}$. But,

as (I) fails, for $N = N_0$, $\exists n' \geq N_0$ s.t. $b_{n'} \leq \frac{r}{2}$. These two

cannot both be true. Hence Case 1 cannot occur.

Hence $b_{N_0} < -r$. Then, $\forall n \geq N_0$, $b_n < b_{N_0} + \frac{r}{2} < -r + \frac{r}{2}$

$=$

i.e. $\forall n \geq N_0$, $b_n < -\frac{r}{2} < 0$

i.e. (II) holds.

Hence either (I) or (II) holds ——— ☒

Back to $\{b_n \mid n \geq 1\}$ Cauchy, $|b_n| > r > 0$ $\forall n \geq 1$

$\Rightarrow \{\frac{1}{b_n} \mid n \geq 1\}$ Cauchy. — Imitate pf of (3) in 1.5.

Remark ① We really do need the condition in (3) (Cauchy version.) eg Let $a_n = 1$, $\forall n$ ($\{a_n\}_{n \geq 1}$ is Cauchy!), $b_n = \frac{1}{n}$, $n \geq 1$.

Then $b_n \to 0$, so $\{b_n\}_{n \geq 1}$ is Cauchy, but $\frac{1}{b_n} = n$, and
$\{n \mid n \geq 1\}$ isn't Cauchy since it isn't bounded!

~~② The lemma we proved has a counterpart: All over~~ →

② These results (1.5 + Cauchy version; bddness of conv seq + bdd'ness of Cauchy sequens) are suggestive: Cauchy sequences share many properties of convergent sequences.

Consider this: sum of Cauchy seq is Cauchy
prod. of " " " " (i.e. their rat'l versions)

Can add & multiply Cauchy sequences! They will, with some appropriate care, be "made into" the real numbers $\underline{\mathbb{R}}$.

Constructing $\mathbb{R}$ (P. 176)

1st. Idea: real numbers are Cauchy sequences of rat'l numbers (their "limit", which shd exist.

Pblm: 2 different Cauchy sequences may represent the same real number — or even rational number (remember: we want $\mathbb{Q} \subset \mathbb{R}$) $\mathbb{R}$

e.g. a_n = n^{th} decimal expansion of $\sqrt{2}$
$a_n' = a_n + \frac{1}{n}$.

Both are Cauchy, and "will both represent $\sqrt{2}$."

| $b_n = 0, \forall n \geq 1$
| $b_n = \frac{1}{n}, \forall n \geq 1$.
"0"

Sol^n to pblm: Def^n Cauchy sequences $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ are equivalent ("will represent same real number") ("will be equal as real numbers") written $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$,

L3
L4 if $\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow |a_n - b_n| < \varepsilon$.
$N = N(\varepsilon)$

(Exercise: suppose $a_n \to a$, $b_n \to b$ and $\{a_n\} \sim \{b_n\}$. Prove that $a = b$.)

Let $A = \{a_n\}_{n\geq 1}$, $B = \{b_n\}_{n\geq 1}$, $C = \{c_n\}_{n\geq 1}$ be 3 any Cauchy sequences of (rational) numbers.

Then ① $A \sim A$ ($|a_n - a_n| = 0 < \varepsilon$, $\forall \varepsilon > 0$, $\forall n \geq 1$, $N = 1$.)

② $A \sim B \iff B \sim A$ ($|a_n - b_n| = |b_n - a_n|$: same $N = N(\varepsilon)$)

③ $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

$$\left(|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq \underbrace{|a_n - b_n|}_{\frac{\varepsilon}{2}} + \underbrace{|b_n - c_n|}_{\frac{\varepsilon}{2}} \right)$$

$\max\{N_a(\frac{\varepsilon}{2}), N_b(\frac{\varepsilon}{2})\}$

So, "$\sim$" enjoys very similar properties to "=". (It is an "equivalence rel^n")

Def^n If A is a Cauchy sequence, define $[A] = \{B \mid B$ is Cauchy, $B \sim A\}$.

Lemma: $[A] = [B] \iff A \sim B$.
For Cauchy A, B

Pf of lemma (<u>Exercise</u>, DGD) Suppose $[A]=[B]$. Since $A\sim A$, $A\in[A]=[B]$, so $A\in[B]$, and hence $A\sim B$. Now suppose $A\sim B$. We show $[A]\subseteq[B]$: Let $C\in[A]$, so $C\sim A$. But $A\sim B$ so $C\sim B$ and hence $C\in[B]$. Thus $[A]\subseteq[B]$. Since $B\sim A$ also holds if $A\sim B$ the same argument shows $[B]\subseteq[A]$. Hence $[A]=[B]$. $\boxed{E}$ (12)

<u>Def^n</u> $\mathbb{R}=\{[A]\mid A$ is a Cauchy sequence of rational numbers$\}$

Want 1) $\mathbb{Q}\subset\mathbb{R}$, ... somehow

2) $\mathbb{R}$ to be a field, with $\mathbb{Q}$ as a subfield

3) $\mathbb{R}$ to be an ordered field, " ordered "

4) so $|\,|:\mathbb{R}\to\mathbb{R}^+$ satisfies all same rules as $|\,|:\mathbb{Q}\to\mathbb{Q}^+$. ..

<u>Bonus, payoff:</u> we don't have to do this again!

<u>Theorem (1.8) p175 (Cauchy, 1821)</u>

Every Cauchy sequence of real numbers converges to a real number!

Indeed, for sequences of real numbers, the sequence is Cauchy iff it converges!

(Already seen pf of conv. $\Rightarrow$ Cauchy, so new part is Cauchy $\Rightarrow$ convergent.

<u>Step 0</u>

Arithmetic operations on $\mathbb{R}$: $A=\{a_n\}_{n\geq1}$, $B=\{b_n\}_{n\geq1}$ Cauchy rational.

$[A]+[B]:=[A+B]$; $A+B=\{a_n+b_n\}_{n\geq1}$ (<u>is</u> Cauchy!)

$[A][B]:=[AB]$ $AB=\{a_nb_n\}_{n\geq1}$ (<u>is</u> Cauchy!)

Zero? $\tilde{0} := [(0,0,\dots)]$ i.e. $\tilde{0} = [\{z_n\}_{n\geq 1}]$, $z_n = 0$, $\forall n \geq 1$.

Negatives! $-[A] = [-A]$, $\left(-\{a_n\}_{n\geq 1} := \{-a_n\}_{n\geq 1}\right)$

One? $\tilde{1} := [(1,1,\dots)]$ i.e. $\tilde{1} = [\{w_n\}_{n\geq 1}]$, $w_n = 1$, $\forall n \geq 1$

Suggest how to get $\mathbb{Q}$ inside $\mathbb{R}$: (real number)

If $q \in \mathbb{Q}$, the real number $\tilde{q} := [(q,q,q,\dots)]$.

Multiplicative Inverses? What does $[A] \neq \tilde{0} = [(0,0,\dots)]$ mean?

$[\{a_n\}_{n\geq 1}] \neq [(0,0,\dots)]$. Since $[A] = [B] \Leftrightarrow A \sim B$, $[A] \neq [B] \Leftrightarrow A \nsim B$.

$\therefore \{a_n\}_{n\geq 1} \nsim \{0\}_{n\geq 1}$. Hence $\exists b > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|a_n - 0| \geq b > 0$

We show $\exists M$ s.t. $\forall p \geq M$, $|a_p| > \frac{b}{2}$:

Since $\{a_n\}_{n\geq 1}$ is Cauchy, for $\varepsilon = \frac{b}{2}$, $\exists N_0$ s.t. $\forall p, m \geq N_0$, $|a_p - a_m| < \frac{b}{2}$

Since $\{a_n\}_{n\geq 1} \nsim \{0\}$, for $N = N_0$, choose $n_0 \geq N_0$ s.t. $|a_{n_0}| \geq b$.

Note: $\forall p \geq N_0$, $|a_p - a_{n_0}| < \frac{b}{2}$ $\quad (-|a_p - a_{n_0}| > -\frac{b}{2}$

Then, $\forall p \geq N_0$,

$$\begin{aligned} |a_{n_0}| &= |a_{n_0} - a_p + a_p| \\ &\leq |a_{n_0} - a_p| + |a_p| \end{aligned}$$

i.e. $\forall p \geq N_0$, $\quad |a_p| \geq |a_{n_0}| - |a_{n_0} - a_p| > b - \frac{b}{2} = \frac{b}{2}$.

$(M = N_0!)$

Theorem: $\alpha = [\{a_n\}_{n\geq 0}] \neq \tilde{0} \Leftrightarrow \exists r > 0$, $\exists N$ s.t. $\forall p \geq N$, $|a_p| > r$.

(Exercise: show that $\alpha \neq \tilde{0} \Leftrightarrow \exists r > 0$, $\exists N$ s.t. Either

1) $\forall p \geq N$, $a_p > r > 0$

or 2) $\forall p \geq N$, $a_p < -r < 0$.

Pf. If $a_{n_0} > 0$, $|a_{n_0}| \geq b \Leftrightarrow a_{n_0} \geq b$ so $a_p \in (a_{n_0} - \frac{b}{2}, a_{n_0} + \frac{b}{2})$

$\Rightarrow a_p > a_{n_0} - \frac{b}{2} \geq \frac{b}{2}$. $\therefore a_p > \frac{b}{2}$. The case $a_{n_0} < 0$ is handled in a similar fashion. (Try it for yourself.)