Mat 2120 F09 Analysis I

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

familiar but... what are 'real' numbers?
("Equivalence classes of Cauchy Sequences of rational numbers")

III. 1

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \}$ $\Big/ \frac{m}{n} = \frac{m'}{n'} \Leftrightarrow mn' = m'n$ $\Big/ \{(m,n) \mid m, n \in \mathbb{Z}, n \neq 0\} \big/ \sim$

$(m_1, n_1) \sim (m_2, n_2) \Leftrightarrow m_1 n_2 = m$

$\mathbb{R} = \mathbb{Q} \cup \{ ? \}$

$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R};\ i^2 = -1 \}$

We will construct $\mathbb{R}$ from $\mathbb{Q}$ in a DGD soon. For the present, consider the properties of $\mathbb{Q}$: 2 operations + order

Addition $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$

$(a, b) \mapsto a + b \quad (= \text{"}+(a,b)\text{"})$

Properties
1) $a + b = b + a$ comm.
2) $(a+b)+c = a+(b+c)$ assoc.
3) $0 + a = a + 0$, $\forall a \in \mathbb{Q}$ $\quad (0 = \frac{0}{n}$
4) $\forall a \in \mathbb{Q}\ \exists b \in \mathbb{Q}$ s.t. $a + b = 0$.
("$b = -a$" : ex. $-a$ is unique)

Mult^n $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$

$(a, b) \longmapsto ab$

Properties
1) $ab = ba$
2) $(ab)c = a(bc)$
3) $1 . a = a$, $\forall a \in \mathbb{Q}$
4) $\forall a \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\exists b \in \mathbb{Q}$ s.t. $ab = 1$.
We write $b = \frac{1}{a}$.

5) $a(b+c) = ab + ac$.

Summary: $\mathbb{Q}$ is a (commutative) field.

$\mathbb{Q}$ is also ordered: ($n \geq m \iff n - m \in \mathbb{N}$. We write "$l \geq 0$" if $l \in \mathbb{Z} \cap \mathbb{N}$; $l > 0$ if $l \geq 0$ & $l \neq 0$.) $n, m \in \mathbb{Z}$

Let $\mathbb{Q}^+ = \{ \frac{m}{n} \in \mathbb{Q} \mid mn \geq 0 \}$

$\mathbb{Q}^- = \{ \frac{m}{n} \in \mathbb{Q} \mid mn < 0 \}$

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$; $\mathbb{Q}^+ \cap \mathbb{Q}^- = \phi$.

Def^n If $a, b \in \mathbb{Q}$, $a \leq b$ iff $b - a \in \mathbb{Q}^+$

(or "$b \geq a$"; $0 < a \iff 0 \leq a$ & $a \neq 0$.)

Facts 1. $\forall a, b \in \mathbb{Q}$, either $a \leq b$ or $b \leq a$.

($a, b, c \in \mathbb{Q}$) 2. If $a \leq b$ and $b \leq a$ then $a = b$

3. If $a \leq b$ and $b \leq c$ then $a \leq c$

(4. $\forall a \in \mathbb{Q}$, $a \leq a$)

5. If $a \leq b$, then $\forall c \in \mathbb{Q}$, $a + c \leq b + c$

6. If $a \leq b$ and (I) $0 \geq c$, then $ac \leq bc$

(II) $c \leq 0$, then $ac \geq bc$

Pf: exercise.

$\mathbb{Q}$ is an ordered field

Def^n If $a \in \mathbb{Q}$, $|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a \leq 0. \end{cases}$

Rmk $\forall a \in \mathbb{Q}$, $|a| \geq 0$.

Proposition $\forall a, b \in \mathbb{Q}$ 1. $|a| = |-a|$

2. $|a| = 0 \iff a = 0$

3. $|a + b| \leq |a| + |b|$. "Δ inequality"

Pf: Exercise

Exercise: If $\forall \varepsilon > 0$, $|a| < \varepsilon$, then $a = 0$.

Rmk (i) The function $|\ |: \mathbb{Q} \to \mathbb{Q}^+$ is called the "absolute value", "modulus" or "norm".

Remark ③ We are familiar with "$\sqrt{2}$".

- Fact $\sqrt{2} \notin \mathbb{Q}$.

— Proof: Suppose $\frac{m}{n} \in \mathbb{Q}$ is such that ($m, n \in \mathbb{Z}$ and:)

(gcd = "greatest common divisor")

(a) $\gcd(m,n) = 1$

(b) $\left(\frac{m}{n}\right)^2 = 2$.

Then $m^2 = 2n^2$. Hence m^2 is even, and so m must be also. Thus $m = 2m_1$ for $m_1 \in \mathbb{Z}$. Then

$$(2m_1)^2 = 2n^2 \Rightarrow 4m_1^2 = 2n^2 \Rightarrow 2m_1^2 = n^2.$$

But then n is also even and this contradicts (a). Hence

$q^2 \neq 2, \ \forall q \in \mathbb{Q}$. $\boxtimes$ —

- — So, what is "$\sqrt{2}$", exactly? It is one of the (very) many "real" numbers. We will consider the actual construction of $\mathbb{R}$ (from $\mathbb{Q}$) soon in a DAP, but for the moment, let's agree that ① $\mathbb{Q} \subset \mathbb{R}$, ② $\mathbb{R}$ is an ordered field with operations add'n, mult'n, and the same 0, 1 as in $\mathbb{Q}$; ③ operations of $\mathbb{R}$, when applied to rational numbers, yield same answers as before ④ The order on $\mathbb{R}$, when applied to $\mathbb{Q} \subset \mathbb{R}$, yields same answers. ⑤ We can define a (norm) absolute value $|\ |: \mathbb{R} \to \mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ with same properties as in the proposition for $\mathbb{Q}$, and that it gives the same results when applied to $\mathbb{Q} \subset \mathbb{R}$.

- Remark Since $\sqrt{2} \in \mathbb{Q}$, but (as we will see shortly)

$$\sqrt{2} \in \mathbb{R},$$

$$\mathbb{Q} \subsetneq \mathbb{R}.$$

(Indeed $\mathbb{R}$ is much "bigger" than $\mathbb{Q}$.)

Limits, Cauchy & Convergent Sequences

Def^n A sequence of rational (or real) numbers is a function $a : \mathbb{N} \setminus \{0\} \to \mathbb{Q}$
$(a : \mathbb{N} \setminus \{0\} \to \mathbb{R})$

- We will write $a(1), a(2), \ldots$ as $a_1, a_2, a_3 \ldots$
- Sometime the domain of the sequence will be all of $\mathbb{N}$.
- We will often denote the sequence by its range, for simplicity, i.e. $\{a_n \mid n \geq 1\}$ or simply $\{a_n\}_{n\geq 1}$.

e.g. $a_n = \frac{1}{n}$, $n \geq 1$. ... $\{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots$

$b_n = 1 + \frac{1}{2^n}$, $n \geq 1$ ... $\{1 + \frac{1}{2^n} \mid n \geq 1\}$

$c_n = \frac{n}{n+1}$, $n \geq 1$ ... $\{c_n \mid n \geq 1\} = \{$

$e_n = \frac{(-1)^n}{n^2}$, $n \geq 1$

$f_n = (-1)^n$, $n \geq 1$.

(recursive def^n:) $r_1 = 1$, $r_{n+1} = \dfrac{r_n^2 + 2}{2r_n}$, $n \geq 2$. $\left(\frac{r_n}{2} + \frac{1}{r_n}\right)$

Ex. 1) r_n is defined for $n \geq 2$! i.e. $r_n \neq 0$.
2) $r_n \in \mathbb{Q}$, $\forall n \geq 1$
3) Think about "$\lim_{n\to\infty} r_n^2$"

Important Def^n Let $\{a_n \mid n \geq 1\}$ be a sequence. A number a is a limit of the sequence ("limit point of the sequence") if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |a - a_n| < \varepsilon$$

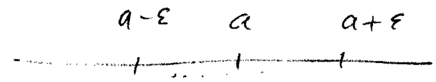
or $\forall \varepsilon > 0$, $n \geq N \Rightarrow |a - a_n| < \varepsilon$.

"You can guarantee that $|a - a_n|$ is as small as you wish as long as n is large

We write " $a = \lim_{n\to\infty} a_n$ " ⑤

We say " the limit of the a_n exists and is a "

or " $\lim_{n\to\infty} a_n$ exists and is a "



or " the <u>sequence $\{a_n \mid n \geq 1\}$ converges to a</u> "

If there is <u>no</u> a with the above property, we say the sequence <u>diverges</u> (or "does not converge")

e.g. $a_n = \frac{n}{n+1}$, $n \geq 1$. <u>Claim</u> $\lim_{n\to\infty} a_n = 1$.

L1 09
L2

Pf. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t.

$N > \frac{1}{\varepsilon} - 1$.

... think...

$\left|1 - \frac{n}{n+1}\right| = \left|\frac{n+1-n}{n+1}\right|$
$= \frac{1}{n+1}$; $\frac{1}{n+1} < \varepsilon \iff$
$\frac{1}{\varepsilon} < n+1$ or $n > \frac{1}{\varepsilon} - 1$

Now suppose $n \geq N$. Then $n > \frac{1}{\varepsilon} - 1$

so $n+1 > \frac{1}{\varepsilon}$, and hence $\varepsilon > \frac{1}{n+1}$.

But then, $n \geq N \Rightarrow \left|1 - \frac{1}{n+1}\right| < \varepsilon$. Hence $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

e.g. $\lim_{n\to\infty} (-1)^n$ does <u>not</u> exist.

Pf (by contradiction) Suppose the limit did exist, and is a. Choose $\varepsilon = \frac{1}{2}$, and let N be the natural number guaranteed for this ε because $\lim_{n\to\infty} (-1)^n = a$. Hence, $\forall n \geq N$, $|a - (-1)^n| < \frac{1}{2}$. Choose $n_0 \geq N$ and n_0 even. (OK!) Then $|a - (-1)^{n_0}| < \frac{1}{2}$ or $|a - 1| < \frac{1}{2}$. Then, $n_0 + 1$ is odd, and $n_0 + 1 \geq N$, so $|a - (-1)^{n_0+1}| < \frac{1}{2}$ as well. That is, $|a+1| < \frac{1}{2}$. But $|a-1| < \frac{1}{2} \iff a \in (\frac{1}{2}, \frac{3}{2})$ and $|a+1| < \frac{1}{2} \iff a \in (-\frac{3}{2}, -\frac{1}{2})$. So $a \in (-\frac{3}{2}, -\frac{1}{2}) \cap (\frac{1}{2}, \frac{3}{2}) = \emptyset$, so this is impossible. Hence, $\lim_{n\to\infty} (-1)^n$ does not exist.

<u>Prop^n</u> If $\lim_{n\to\infty} a_n$ exists, it is unique!

Pf. Suppose not, and that $\lim_{n\to\infty} a_n = a \neq b = \lim_{n\to\infty} a_n$.

Let $d = |a-b|$. Note that $d > 0$. Let $\varepsilon = \frac{d}{2}$

Since $\lim_{n\to\infty} a_n = a$, $\exists N_a$ s.t. $\forall n \geq N_a$, $|a - a_n| < \varepsilon$

$\lim_{n\to\infty} a_n = b$, $\exists N_b$ s.t. $\forall n \geq N_b$, $|b - a_n| < \varepsilon$.

Now choose $N = \max\{N_a, N_b\}$. Then $N \geq N_a$, $N \geq N_b$ and so $|a - a_N| < \varepsilon$ <u>and</u> $|b - a_N| < \varepsilon$. But then

$$|a-b| = |a - a_n + a_n - b| \leq |a-a_n| + |a_n - b| < \varepsilon + \varepsilon = 2\varepsilon = d$$

i.e. $d = |a-b| < d$! This is a contradiction. Hence $a = b$. $\square$

<u>Remark</u> If $\{a_n\}_{n\geq 1}$ is convergent (say, to a), the a_n "get closer and closer to each other"; precisely, given $\varepsilon > 0$, (thinking of $\varepsilon' = \frac{\varepsilon}{2}$) we may choose N s.t. $n \geq N \Rightarrow |a - a_n| < \frac{\varepsilon}{2}$. If $n, m \geq N$ then

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon.$$

<u>Def^n</u> A sequence $\{a_n | n \geq 1\}$ is a <u>Cauchy</u> sequence if

$$\forall \varepsilon > 0, \quad \exists N \text{ s.t. } \quad m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon$$

$$(\forall \varepsilon > 0, \quad \exists N \text{ s.t. } \forall m, n \geq N, \quad |a_m - a_n| < \varepsilon.)$$

Here is no reference to a limit.

<u>FACT</u>: Every convergent sequence is Cauchy. (We proved this in our previous remark).

<u>QUESTION</u>: Does every Cauchy sequence converge?

A: 1) No! (not every Cauchy sequence of rat'ls has a rat'l limit,
2) Yes!! (every Cauchy sequence of real numbers has a real limit

e.g. (for (1)). $\sqrt{2} = 1.41421356\ldots$; $a_1 = 1.4 = \frac{14}{10}$, $a_2 = 1.41 = \frac{141}{100}$, $a_3 = 1.414 = \frac{1414}{1000}$ ...

Let a_n = decimal expansion of $\sqrt{2}$ up to and including n^{th} decimal place.

If $n, m \geq N$, $|a_n - a_m| \leq \frac{9.\dot{9}}{10^{N+1}} = \frac{1}{10^N}$ (a_n, a_m differ at most in the $N+1^{st}$ decimal place

Now let $\varepsilon > 0$, and choose N s.t. $\frac{1}{10^N} < \varepsilon$ (Easy: we know from this mornings DHD that $10^n > 2^n \geq n$ ($\forall n \geq 1$), so simply choose $N > \frac{1}{\varepsilon}$. Then $10^N > \frac{1}{\varepsilon}$.) Hence, $\forall \varepsilon > 0$ $\exists N$ s.t.

$m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon$. Hence this sequence is Cauchy.

However, its limit is surely $\sqrt{2}$, which is not rational...

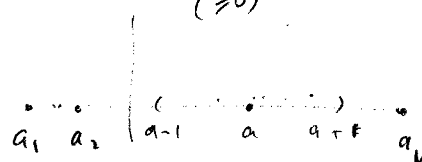
(There is an explicit construction of a rat'l sequence that you can prove is Cauchy and which must converge to $\sqrt{2}$. See me for details.)

"Yes" - we need to wait for the construction of the reals!

Some facts about convergent sequences:

L2
L3 **Thm 1.3 (P.72)** If $\{a_n\}_{n \geq 1}$ converges, it is bounded, i.e. $\exists B_{(\geq 0)}$ s.t.

$\forall n$, $|a_n| \leq B$. (i.e. $-B \leq a_n \leq B$



Pf. Suppose $\lim_{n \to \infty} a_n = a$. For $\varepsilon = 1$, $\exists N$ s.t. $n \geq N \Rightarrow$

$|a - a_n| < 1$. Hence $a_n \in (a-1, a+1)$, if $n \geq N$.

or, $|a_n| = |a_n - a + a| \leq 1 + |a|$, if $n \geq N$.

Now let $B = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a|\}$.

Then, $\forall n \geq 1$, $|a_n| \leq B$. ☐

Remark The converse is false: not every bdd sequence converges: $\{(-1)^n\}_{n \geq 1}$ is certainly bdd ($|(-1)^n| \leq 1$, $\forall n$)
but it does not converge