TA

1. Let $a, b, x, y \in \mathbf{R}$. Prove that:

3 a) $\max\{x,y\} = \frac{|x-y|}{2} + \frac{x+y}{2}$.

4 b) $|x| > b > 0 \Rightarrow |\frac{1}{a} - \frac{1}{x}| \le \frac{|a-x|}{b|a|}$. $(a \neq 0!)$

1 - using cases or "symmetry"

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2

a) If $x \le y$, $\max\{x,y\} = y$, while $|x-y| = y-x$

$\therefore \frac{|x-y|}{2} + \frac{x+y}{2} = \frac{y-x}{2} + \frac{x+y}{2} = \frac{y}{2} + \frac{y}{2} = y = \max\{x,y\}$

If $x > y$, $\max\{x,y\} = x$, while $|x-y| = x-y$.

Hence, $\frac{|x-y|}{2} + \frac{x+y}{2} = \frac{x-y}{2} + \frac{x+y}{2} = \frac{x}{2} + \frac{x}{2} = x = \max\{x,y\}$

b) $\left|\frac{1}{a} - \frac{1}{x}\right| = \left|\frac{x-a}{ax}\right| =$ 1 $\frac{|x-a|}{|a|\,|x|}$. But $|x| > b > 0$

$\therefore \frac{1}{|x|} < \frac{1}{b} \therefore \frac{|x-a|}{|x|\,|a|} < \frac{|x-a|}{b.|a|}$, as req'd.

2

1 realizing that $x < y \Rightarrow x \le y$

0, 1, or 3

2. a) If $\{a_n\}_{n\geq 1}$ is a real sequence and $a \in \mathbf{R}$, give the definition of "$\lim_{n\to\infty} a_n = a$." 3

b) Suppose $a_n = \frac{n^2+1}{3(n+1)^2}, n \in \mathbf{N}$. Find $\lim_{n\to\infty} a_n$, and prove carefully that it exists, using the definition.

a) $\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n \geq N, |a - a_n| < \varepsilon$.

b) We claim $\lim_{n\to\infty} a_n = \frac{1}{3}$: $\left|\frac{1}{3} - a_n\right| = \left|\frac{1}{3} - \frac{n^2+1}{3(n+1)^2}\right|$

$= \left|\frac{n^2+2n+1-(n^2+1)}{3(n+1)^2}\right| = \frac{2n}{3(n+1)^2} < \frac{2n}{3n^2}$ (Since $(n+1)^2 > n^2$)

$= \frac{2}{3n}$. Let $\varepsilon > 0$, and choose $N > \frac{2}{3\varepsilon}$. Then,

$\forall n \geq N$, $n > \frac{2}{3\varepsilon}$, so $\frac{2}{3n} < \varepsilon$. Hence, $\forall n \geq N$,

$\left|\frac{1}{3} - a_n\right| < \varepsilon$. Thus, $\lim_{n\to\infty} a_n = \frac{1}{3}$

1 - $\frac{1}{3}$

1 - $\left|\frac{1}{3} - a_n\right| < b_n$, with $b_n \to 0$

1 - $b_n < c\frac{1}{n}$ somehow (well-written)

1 Good choice of N

TA

3. a) If $\{c_n\}_{n\geq 1}$ is a real sequence, give the definition of "$\{c_n\}_{n\geq 1}$ is a Cauchy sequence." 2

b) Suppose $\{a_n\}_{n\geq 1}$,is a Cauchy sequence. Prove that if $\exists r > 0$ s.t. $\forall n \in \mathbf{N}$, $a_n > r$, then $\{\frac{1}{a_n}\}_{n\geq 1}$ is a Cauchy sequence. 0 or 2

a) $\{C_n\}_{n\geq 1}$ is Cauchy if, $\forall \varepsilon > 0$, $\exists N$ s.t. $m, n \geq N$
$\Rightarrow |C_n - C_m| < \varepsilon$.

b) Let $n, m \in \mathbb{N}$. Then $\left|\frac{1}{a_n} - \frac{1}{a_m}\right| = \left|\frac{a_m - a_n}{a_n a_m}\right| = \frac{|a_m - a_n|}{a_n a_m}$

But $a_n > r$, $a_m > r$, so $\frac{1}{a_n a_m} < \frac{1}{r^2}$. Hence

$$\left|\frac{1}{a_n} - \frac{1}{a_m}\right| < \frac{|a_m - a_n|}{r^2}$$

Let $\varepsilon > 0$, choose N s.t. $\forall n, m \geq N$,
$|a_n - a_m| < \varepsilon \cdot r^2$. Then, $\forall n, m \geq N$

$$\left|\frac{1}{a_n} - \frac{1}{a_m}\right| < \frac{\varepsilon r^2}{r^2} = \varepsilon.$$

Thus, $\{\frac{1}{a_n}\}_{n\geq 1}$ is Cauchy

1 $\rightarrow \frac{|a_m - a_n|}{a_m a_n}$

2 establishing $\cdots < \frac{|a_m - a_n|}{r^2}$

1 Good choice of N

1 Well-written

0 of 2

2

5. a) Define what is meant by an "accumulation point" of a real sequence $\{x_n\}_{n\geq 1}$.

b) Let $\{x_n\}_{n\geq 1}$ be a bounded real sequence and let

$$x := \sup\{x_n \mid n \geq 1\}.$$

Suppose that $\forall n \in \mathbf{N}, x_n < x$. Prove that x is an accumulation point of $\{x_n\}_{n\geq 1}$.

a) An accumulation point of $\{x_n\}_{n\geq 1}$ is the limit of some convergent subsequence of $\{x_n\}_{n\geq 1}$.

b) Let $k \in \mathbb{N}$, $k \geq 1$, and define $S_k = \{n \in \mathbb{N} \mid x_n \in (x - \frac{1}{k}, x)\}$

Each S_k is non-empty, and in fact has ∞ly many elements: if not, let $y = \max S_k$. Then $y = x_N$ for some $N \in \mathbb{N}$, and so $y < x$. But then, $\forall n \in \mathbb{N}$, $x_n \leq y$ (since $y = \max S_k$), and so $\sup\{x_n \mid n\geq 1\} = \max\{x_n \mid n \geq 1\} = y$, contradicting $y < x$.

Now choose $n_1 \in S_1$; Since S_2 is infinite, $\exists n_2 \in S_2$ with $n_2 > n_1$.
Having chosen $n_j \in S_j$ with $n_j > n_{j-1}$ for $j = 2, \ldots, k-1$, we choose $n_{k+1} \in S_{k+1}$ with $n_{k+1} > n_k$. This is possible since S_{k+1} is infinite.
Hence, we have a subsequence $\{x_{n_k}\}_{k\geq 1}$ with the property that $x_{n_k} \in (x - \frac{1}{k}, x)$, since $n_k \in S_k$. So let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ s.t. $M > \frac{1}{\varepsilon}$. Then if $k \geq M$,

$\frac{1}{k} \leq \frac{1}{M} < \varepsilon$, and so $|x - x_{n_k}| < \frac{1}{k} < \varepsilon$.

Hence, $\lim_{k\to\infty} x_{n_k} = x$, so x is an accumulation point of $\{x_n\}_{n\geq 1}$.

1 "S_k is infinite"

2 proof of *

1 "$n_{k+1} > n_k$"

1 $x_{n_k} \to x$

(TA) 0, 1 or 3

4 a) Let $\{t_n\}_{n\geq 1}$ be a real sequence. Define what is meant by "the series $\sum_{n\geq 1} t_n$ converges." 3

b) Does the series $\sum_{n\geq 1} \frac{n+1}{n^3+1}$ converge?

a) Let $S_n = \sum_{j=1}^{n} t_j$. Then $\sum_{n\geq 1} t_n$ <u>converges</u> if $\lim_{n\to\infty} S_n$ exists.

b) Note that $n^3+1 > n^3$, $\forall n \in \mathbb{N}$, and $n+1 \leq 2n$, $\forall n \in \mathbb{N}$.

Hence $\frac{n+1}{n^3+1} < \frac{2n}{n^3} = \frac{2}{n^2}$.

But we ~~know~~ that (from class) $\sum_{n\geq 1} \frac{1}{n^2}$ converges. So too, then, does $\sum_{n\geq 1} \frac{2}{n^2}$. By comparison (all terms are +ve), $\sum_{n\geq 1} \frac{n+1}{n^3+1}$ also converges.

1 - trying comparison

2 - good estimate that <u>will</u> work + proving the estimate

1 - well written

6. (Bonus) $(3\frac{1}{2}\text{pts})$

2 a) Let $\{a_n\}_{n\geq 1}$ be a decreasing, positive sequence such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} na_n = 0$.

$1\frac{1}{2}$ b) Suppose $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

a) Let $\varepsilon > 0$, and choose N s.t. $\forall n \geq N$, $\forall k$, $a_{n+1} + \quad + a_{n+k} < \frac{\varepsilon}{2}$.

But $a_{n+j} \geq a_{n+k}$, $j = 1, .. k$, so $k.(a_{n+k}) < \frac{\varepsilon}{2}$ if $n \geq N$ and $k \geq 1$.

Hence, if $k \geq N$ and $n \geq 1$, $n.(a_{k+n}) < \frac{\varepsilon}{2}$. Thus if $k, n \geq N$,

$(n+k)a_{n+k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Let $M = 2N$, and choose $m \geq M$. Write

$m = n+k$ with $n \geq N$, $k \geq N$. Then, $0 \leq m a_m = (n+k)a_{n+k} < \varepsilon$.

Hence $\lim_{m\to\infty} m a_m = 0$

b) Note that if a, b are real, $0 \leq (|a| - |b|)^2 = a^2 - 2|ab| + b^2$.

Hence, $|ab| \leq \frac{1}{2}(a^2 + b^2)$. The series $\sum_{n=1}^{\infty} \frac{1}{2}(a_n^2 + b_n^2)$ converges

because $\sum a_n^2$ & $\sum b_n^2$ do. Hence $\sum_{n\geq 1} |a_n b_n|$ converges.

Thus, $\sum_{n\geq 1} a_n b_n$ also converges.