2. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Prove that $\dfrac{\partial f}{\partial w}(0,0)$ exists for all directions $w \in \mathbf{R}^2, \|w\| = 1$, but that f is not differentiable at $(0,0)$.

Suppose $w = (w_1, w_2)$ satisfies $\|w\| = 1$. Then $\frac{\partial f}{\partial w}(0,0) =$

$$\lim_{h \to 0} \frac{f(hw_1, hw_2)}{h} = \lim_{h \to 0} w_1^2 w_2 \cdot \frac{h^3}{h^4 w_1^4 + h^2 w_2^2} = \lim_{h \to 0} \frac{w_1^2 w_2 h}{h^2 w_1^4 + w_2^2}$$

$= 0$ (even if $w_2 = 0$, since then $w_1^2 = 1$, and the quotient in the limit is 0.)

Hence, $\frac{\partial f}{\partial w}(0,0) = 0$, $\forall$ directions w.

However, suppose f were differentiable at $(0,0)$. Then $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ by the previous calculation, so $\exists\, r : U \to \mathbb{R}$ (U an open set containing $(0,0)$) s.t. ① $f(x,y) = r(x,y)\|(x,y)\|$ and

② $\lim_{(x,y)\to(0,0)} r(x,y) = r(0,0) = 0$.

But ① $\Rightarrow$ $r(x,y) = \dfrac{f(x,y)}{\|(x,y)\|} = \dfrac{x^2 y}{(x^4+y^2)\sqrt{x^2+y^2}}$.

Let $c \in \mathbb{R}$ and define $v_n = \left(\frac{1}{n}, \frac{c}{n^2}\right)$. Then $v_n \to 0, \forall c$, but $\lim_{n\to\infty} r(v_n)$

$= \lim_{n\to\infty} \dfrac{c/n^4}{1/n^4 + c^2/n^4} = \dfrac{c}{1+c^2}$. If $c = 0$, this is 0, but if $c = 1$, this is $\frac{1}{2}$.

Hence $\lim_{v\to 0} r(v)$ does not exist. Hence f is not diff'ble at 0.

1. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \sqrt{|xy|}$$

a) Prove that f is continuous at $(0,0)$ but is not differentiable there.

b) Prove that there exist directions $w \in \mathbf{R}^2$, $||w|| = 1$, such that $\dfrac{\partial f}{\partial w}(0,0)$ does not exist.

a) Since $|x| \leq \sqrt{x^2+y^2}$, $|y| \leq \sqrt{x^2+y^2}$, we have

$|f(x,y) - f(0,0)| = |\sqrt{|xy|}| \leq \sqrt{(\sqrt{x^2+y^2})^2} = \sqrt{x^2+y^2}$. Hence f is cts at $(0,0)$ ("$\delta = \varepsilon$") in (6). We will show that a directional derivative at $(0,0)$ does not exist and hence we can conclude that f is not diff'ble at $(0,0)$

b) Let $w = \frac{\sqrt{2}}{2}(1,1)$. Then $\dfrac{\partial f}{\partial w}(0,0) = \lim\limits_{h \to 0} \dfrac{\sqrt{\frac{h^2}{2}}}{h} = \lim\limits_{h \to 0} \frac{\sqrt{2}}{2} \cdot \frac{|h|}{h}$,

which does not exist. (Indeed, unless $w = (1,0)$ or $(0,1)$, $\dfrac{\partial f}{\partial w}(0,0)$ does not exist.)

3. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$, that both partial derivatives exist there, but that f is not differentiable at $(0,0)$.

Since $\displaystyle |f(x,y) - f(0,0)| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\left(\sqrt{x^2+y^2}\right)^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$,

f is cts at $(0,0)$ ("$\delta = \varepsilon$").

Moreover, $\displaystyle \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$; similarly, $\displaystyle \frac{\partial f}{\partial y}(0,0) = 0$

If f were diff'ble at $(0,0)$, all directional derivatives would exist, and be given by the formula $\displaystyle \frac{\partial f}{\partial w}(0,0) = \nabla f(0,0) \cdot w$.

But $\nabla f(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x}(0,0) \\ \frac{\partial f}{\partial y}(0,0) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so all directional derivatives

would be zero. However, if $w = \frac{\sqrt{2}}{2}(1,1)$, $\displaystyle \frac{\partial f}{\partial w}(0,0) = \lim_{h \to 0} \frac{\frac{1}{2}h^2}{h|h|} =$,

$\displaystyle = \lim_{h\to 0} \frac{h}{2|h|}$, which does not exist.

4. (Bonus) Let

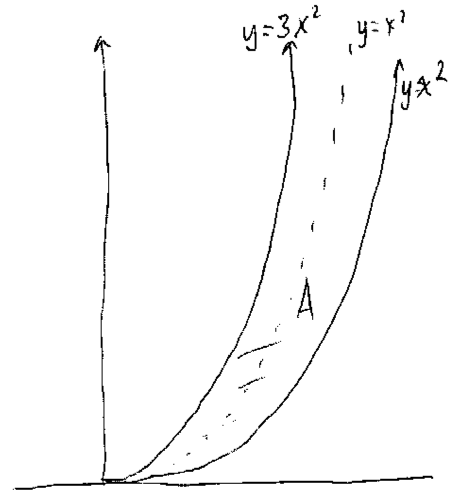
$$A = \{(x, y) \in \mathbf{R}^2 \mid x > 0 \text{ and } x^2 < y < 3x^2\}.$$

Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function such that

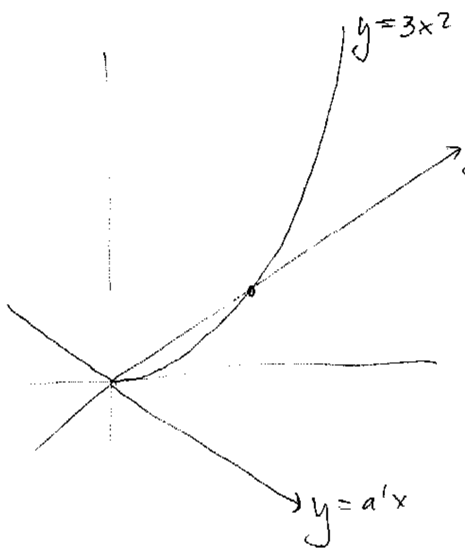
(i) $f(x, y) = 0$ for all $(x, y) \in A^c$,

(ii) $f(x, 2x^2) = x$ for all $x > 0$, and

(i) $0 \le f(x, y) \le x$ for all for all $x > 0, y \in \mathbf{R}$.



Prove that all directional derivatives $\dfrac{\partial f}{\partial w}(0,0)$ exist and are zero, but that f is not differentiable at $(0,0)$



If $-\infty < a' \le 0$, $(x, a'x) \in A^c$ $\forall x > 0$,

So $\dfrac{\partial f}{\partial w}(0,0) = 0$ if $w = \dfrac{(1, a')}{\sqrt{1+(a')^2}}$.

If $0 < a < \infty$, $\forall x < \frac{a}{3}$, $x > 0$, $(x, ax) \in A^c$ as well. Hence $\dfrac{\partial f}{\partial w}(0,0) = 0$ if $w = \dfrac{(1, a)}{\sqrt{1+a^2}}$.

Clearly, $\dfrac{\partial f}{\partial y}(0,0) = 0$ as well

Hence, all directional derivatives exist and are zero.

Suppose f were differentiable at $(0,0)$. Then, as $\nabla f = 0$ from the above, $f(x, x^2) = r(x, x^2)\|(x, x^2)\|$ for some f^n r, cts at $(0,0)$ with $r(0,0) = 0$, and $|x|$ small enough. But $\lim_{x\to 0} r(x, x^2) = \lim_{x \to 0} \dfrac{f(x,x^2)}{\sqrt{x^2+x^4}}$ does not exist, since $\lim_{x\to 0^+} \dfrac{f(x,x^2)}{\sqrt{x^2+x^4}} = \lim_{x \to 0^+} \dfrac{x}{\sqrt{x^2+x^4}} = 1$ while $\lim_{x\to 0^-} \dfrac{f(x,x^2)}{\|(x,x^2)\|} = 0$.

Hence f is not diff'ble at $(0,0)$