

1. a) Let $l \in \mathbf{R}$ and define $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ by $f(x, y) = \begin{cases} l & \text{if } (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$

Prove carefully that there is no value of l for which f is continuous at $(0, 0)$

① idea + ① calc

b) Let $s \in \mathbf{R}$ and define $g : \mathbf{R}^2 \rightarrow \mathbf{R}$ by $g(x, y) = \begin{cases} s & \text{if } (x, y) = (0, 0) \\ \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$

① idea + ① calc

Prove carefully that there is a unique value of s for which g is continuous at $(0, 0)$.

a) Let $v_n = (\frac{1}{n}, 0)$, $u_n = (\frac{1}{n}, \frac{1}{n})$, $n \geq 1, n \in \mathbf{N}$. Then $\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} u_n = 0$.

but $\lim_{n \rightarrow \infty} f(v_n) = 0$ while $\lim_{n \rightarrow \infty} f(u_n) = \frac{1}{2}$. Hence

$\lim_{v \rightarrow 0} f(v)$ does not exist, and so there is no $l \in \mathbf{R}$ that would make f cts at $v = 0$

b) Note that $|g(x, y) - 0| = \left| \frac{x^2 y}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2) \sqrt{x^2 + y^2}}{x^2 + y^2} = \sqrt{x^2 + y^2}$.

Hence $\lim_{v \rightarrow 0} g(v) = 0$ (let $\delta = \epsilon$). Thus, if we set $s = 0$,

g is cts at 0 , and if $s \neq 0$, $\lim_{v \rightarrow 0} g(v) \neq s$, so g is not cts at 0 .

[12.5] Here X^c denotes the complement $\mathbb{R}^n \setminus X$.

2. Let $A \subset \mathbb{R}^n$. A point $b \in \mathbb{R}^n$ is called a *limit point* of A if every open ball centred on b contains at least one point of A . Let $\bar{A} := \{b \mid b \text{ is a limit point of } A\}$ be the set of all limit points of A . Find \bar{A} for the following sets:

\bar{A}

① + ① + ② $\bar{A} \subset A$

a) $\{2\} \cup \{\frac{1}{n} \mid n \geq 1\} \subset \mathbb{R}$.

We first note that $\forall a \in A, \forall r > 0, a \in B(a, r)$, so $A \subseteq \bar{A}$.

$B \subset \bar{A}$

b) $\{(1, 1)\} \subset \mathbb{R}^2$.

c) $\{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x \neq 0\}$.

As well, $b \in \bar{A} \Leftrightarrow \exists$ sequence $\{a_n\}_{n \geq 1} \subset A$ s.t. $\lim_{n \rightarrow \infty} a_n = b$ (Pf: If $b \in \bar{A}$, choose $a_n \in B(b, \frac{1}{n}) \cap A$ of

d) $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

Then $a_n \rightarrow b$. Conversely, suppose $\{a_n\} \subset A$ & $a_n \rightarrow b$. For $r > 0$, choose $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - b| < r$. Then $a_n \in B(b, r) \cap A$, so

e) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}$.

$b \in \bar{A}$.

a) We claim $\bar{A} = A \cup \{0\}$. Since $\frac{1}{n} \in A, \forall n \geq 1$ and $\frac{1}{n} \rightarrow 0, 0 \in \bar{A}$. Note

that $(A \cup \{0\})^c = (-\infty, 0) \cup \bigcup_{n \geq 1} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, 2) \cup (2, \infty)$. If $x < 0$,

$B(x, \frac{|x|}{2}) \cap A = \emptyset$, so $x \notin \bar{A}$; if $x > 2, B(x, \frac{|x-2|}{2}) \cap A = \emptyset$, so $x \notin \bar{A}$.

If $x \in (a, b) \subset (A \cup \{0\})^c$, then $B(x, r_x) \subset (a, b)$ and so $x \notin \bar{A}$.

Hence, $\bar{A} = A \cup \{0\}$.

b) We claim $A = \bar{A}$ here. If $v \neq (1, 1)$, let $r = \|v - (1, 1)\| > 0$ then

$B(v, r) \cap \{(1, 1)\} = \emptyset$, so $v \notin \bar{A}$.

c) Again, we claim $A = \bar{A}$. Note that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ def'd by $f(x, y) = xy$ is cts, and $A = f^{-1}(\{1\})$. But $\{1\}^c = (-\infty, 1) \cup (1, \infty)$, which is open, so f^{-1} is closed. Hence A is closed. Thus any sequence in A which converges,

converges in A . Hence $\bar{A} \subseteq A$, and so $A = \bar{A}$.

d) Let $f(x, y) = xy$ again, and note that $A = f^{-1}(\{0\})$. Again $\{0\}$ is closed, and so too is A . Hence $\bar{A} \subseteq A$ and so $A = \bar{A}$.

e) We claim $\bar{A} = B = \{(x, y) \mid x^2 + y^2 \geq 1\}$. If $g(x, y) = x^2 + y^2$, g is cts and since $B = g^{-1}([1, \infty))$, and $[1, \infty)$ is closed, then B is closed, and $B \supseteq A$.

$$A = h^{-1}(B)$$

$$\Leftrightarrow h(A) \subseteq B$$

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3. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuous, $U \subset \mathbb{R}^m$ open and $F \subset \mathbb{R}^m$ closed.

a) Prove that $h^{-1}(U) := \{v \in \mathbb{R}^n \mid h(v) \in U\}$ is open in \mathbb{R}^n .

$$\textcircled{2} = \textcircled{1} \text{ (def.)} \cdot \textcircled{1} \text{ (cont.)}$$

b) Prove that $h^{-1}(F) := \{v \in \mathbb{R}^n \mid h(v) \in F\}$ is closed in \mathbb{R}^n .

$$\textcircled{1} \text{ (def.)} + \textcircled{1} \text{ (cont.)}$$

c) Prove that $\{v \in \mathbb{R}^n \mid h(v) = 0\}$ is closed in \mathbb{R}^n .

$\textcircled{1}$

d) Prove that $S^n = \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$ is compact.

$\textcircled{1}$ (bde) $\textcircled{1}$ closed

a) Let $v \in h^{-1}(U)$. Hence $h(v) \in U$. Let $\varepsilon > 0$ with $B(h(v), \varepsilon) \subset U$. Since h is cont. at v , $\exists \delta > 0$ s.t. $h(B(v, \delta)) \subset B(h(v), \varepsilon) \subset U$. But this is equivalent to $B(v, \delta) \subset h^{-1}(U)$. Hence $h^{-1}(U)$ is open.

b) Since $A = h^{-1}(B) \Leftrightarrow h(A) \subseteq B$, then if $A = h^{-1}(B)$, $x \in A \Leftrightarrow h(x) \in B$, and so $x \notin A \Leftrightarrow h(x) \notin B \Leftrightarrow$

$A \subseteq \mathbb{R}^m$, then $(h^{-1}(A^c))^c = h^{-1}(A)$, as follows:

$x \in h^{-1}(A) \Leftrightarrow h(x) \in A \Leftrightarrow h(x) \notin A^c \Leftrightarrow x \notin h^{-1}(A^c) \Leftrightarrow x \in (h^{-1}(A^c))^c$. Thus, if F is closed, F^c is open,

so by (a) $h^{-1}(F^c)$ is open, so $(h^{-1}(F^c))^c$ is closed,

But $(h^{-1}(F^c))^c = h^{-1}(F)$, so $h^{-1}(F)$ is closed.

c) $\{0\} \subset \mathbb{R}^m$ is closed because if $v \notin \{0\}$, $B(v, \frac{\|v\|}{2}) \cap \{0\} = \emptyset$, and so $\{0\}^c$ is open. By (b) $\{v \mid h(v) = 0\} = h^{-1}(\{0\})$ is closed.

d) $\|\cdot\| : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is continuous, since the Δ inequality shows $|\|x\| - \|y\|| \leq \|x - y\|$. Then, $S^n = (\|\cdot\|)^{-1}(\{1\})$ is closed because $\{1\}$ is closed in \mathbb{R} . As well, S^n is bdd. Thus S^n is compact.

4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.

a) Prove that T is continuous on \mathbb{R}^n iff T is continuous at 0.

b) Deduce from (a) that T is continuous on \mathbb{R}^n iff T is uniformly continuous on \mathbb{R}^n .

For the rest of this question, you may assume that T is continuous on \mathbb{R}^n .

c) Prove that $\exists M \in \mathbb{R}$ such that, $\forall u \in \mathbb{R}^n$,

$$\|T(u)\| \leq M\|u\|.$$

(Hint: Use 3(d) and a theorem from class to first show that $\exists b \geq 0$ such that, $\forall w \in S^{n-1}, \|T(w)\| \leq b$.)

a) It suffices to show that T is cts at 0 $\Rightarrow T$ is cts at $a, \forall a \in \mathbb{R}^n$.
Let $\epsilon > 0$, and choose $\delta > 0$ s.t. $\|u\| < \delta \Rightarrow \|T(u)\| < \epsilon$. Then, if $\|x-a\| < \delta, \|T(x-a)\| < \epsilon$. But $T(x-a) = T(x) - T(a)$, so T is cts at a .

b) The $\delta(\epsilon)$ chosen in (a) for cty at 0 depended only on ϵ , and not on x or a . Hence T is uniformly cts on \mathbb{R}^n .

c) Since T is cts on \mathbb{R}^n , and S^{n-1} is compact, T is bounded on S^{n-1} . i.e. $\exists M > 0$ s.t. $\|T(w)\| \leq M, \forall w \in S^{n-1}$. But

then, if $u \neq 0, u = \|u\| \cdot \frac{u}{\|u\|}$ and $T(u) = \|u\| T\left(\frac{u}{\|u\|}\right)$. Since

$$\left\| \frac{u}{\|u\|} \right\| = 1, \left\| T\left(\frac{u}{\|u\|}\right) \right\| \leq M. \text{ Thus } \|T(u)\| \leq \|u\| \cdot M,$$

$\forall u \neq 0$. For $u = 0$, this also holds since $T(u) = 0$.

Handwritten notes and diagrams:

- Diagram showing $\text{cts at } 0 \Rightarrow \text{cts on } \mathbb{R}^n$ and $\text{cts on } \mathbb{R}^n \Rightarrow \text{cts at } 0$.
- Diagram showing $\text{cts on } \mathbb{R}^n \Rightarrow \text{uniformly cts on } \mathbb{R}^n$.
- Diagram showing $\text{cts on } \mathbb{R}^n \Rightarrow \text{cts on } S^{n-1}$.
- Diagram showing $\text{cts on } S^{n-1} \Rightarrow \exists M$.
- Diagram showing $\text{cts on } S^{n-1} \Rightarrow \text{bounded}$.
- Diagram showing $\text{cts on } S^{n-1} \Rightarrow \exists b$.

5. (Bonus) Let $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and define $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by $T(v) = Av$.

a) Prove carefully that T is continuous on \mathbf{R}^2 .

b) Find $v_{\min}, v_{\max} \in \{v \in \mathbf{R}^2 \mid \|v\| = 1\}$ such that

$$\forall v \in \mathbf{R}^2, \quad \|T(v_{\min})\| \|v\| \leq \|T(v)\| \leq \|T(v_{\max})\| \|v\|.$$

(We know v_{\min}, v_{\max} exist. You are supposed to actually find them, and prove the inequalities above.)

a) By 4(a), it suffices to show T is cts at $(0,0)$. But

$$\|T(x,y)\| = \sqrt{(2x+y)^2 + (x+2y)^2} = \sqrt{5x^2 + 8xy + 5y^2} \leq 3\sqrt{x^2+y^2}$$

(because $2xy \leq x^2 + y^2$, so $8xy \leq 4x^2 + 4y^2$) Hence, $\forall \varepsilon > 0$,
let $\delta = \frac{\varepsilon}{3}$. Then $\|v\| < \delta \Rightarrow \|T(v)\| < \varepsilon$.

b) Note that $\|T(x,y)\|^2 = 9 \left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2$

and that $x^2 + y^2 = 1 \Leftrightarrow \underbrace{\left(\frac{x+y}{\sqrt{2}}\right)^2}_a + \underbrace{\left(\frac{x-y}{\sqrt{2}}\right)^2}_b = 1$.

Hence $\max \{ \|T(v)\|^2 \mid \|v\| = 1 \}$

$$= \max \{ 9a^2 + b^2 \mid a^2 + b^2 = 1 \} \quad \begin{matrix} 9a^2 + (1-a^2) \\ = 8a^2 + 1 \end{matrix}$$

$$= \max \{ 8a^2 + 1 \mid 0 \leq a^2 \leq 1 \}$$

$$= 9, \text{ and this occurs when } a = \pm 1; b = 0$$

Moreover $\min \{ \|T(v)\|^2 \mid \|v\| = 1 \} = \min \{ 8a^2 + 1 \mid 0 \leq a^2 \leq 1 \} = 1$ when
 $a = 0; b = \pm 1$. Hence $v_{\min} = \pm \frac{\sqrt{2}}{2} (1, 1)$ $v_{\max} = \pm \frac{\sqrt{2}}{2} (1, -1)$

v_{\min}
 v_{\max}
(2) just in