1. Define $f : [0,3] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{x-2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2. \end{cases}$$

a) If $a \in [0,3]$, define '*f is continuous at a.*'

b) Using the definition, show carefully that f is continuous at $x = 1$. (i.e use '$\varepsilon - \delta$'.)

c) Show carefully, using the definition, that f is *not* continuous at $x = 2$.

a) f is cts at a if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta$ (and $x \in [0,3]$) $\Rightarrow |f(x) - f(a)| < \varepsilon$.

b) Note that $f(1) = -1$, so $|f(x) - f(1)| = \left|\frac{1}{x-2} + 1\right| = \left|\frac{1 + x - 2}{x-2}\right|$

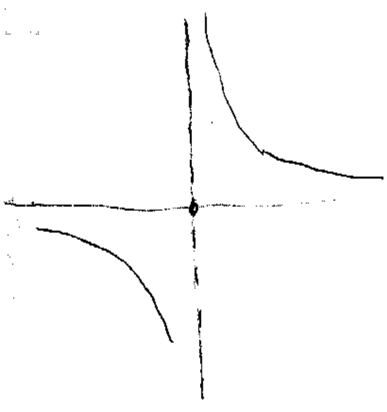
$= \frac{|x-1|}{|x-2|}$. We need an (upper) estimate for $\frac{1}{|x-2|}$, i.e. a lower bound for $|x-2|$. If we restrict x s.t. $|x-1| < \frac{1}{2}$, then $\frac{1}{2} < x < \frac{3}{2}$, so $-\frac{3}{2} < x - 2 < -\frac{1}{2}$, and hence $\frac{3}{2} > |x-2| > \frac{1}{2}$. Thus, $|x-1| < \frac{1}{2} \Rightarrow \frac{1}{|x-2|} < 2$.

So, let $\varepsilon > 0$ and choose $\delta = \frac{1}{2}\min(\varepsilon, 1) = \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\right)$. Then,

$|x-1| < \delta \Rightarrow |f(x) - f(1)| = \frac{|x-1|}{|x-2|} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

Hence, f is cts at $x = 1$.

c) We must show that $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x$ with $|x-2| < \delta$ s.t. $|f(x) - f(2)| \geq \varepsilon$.



Let $\varepsilon = 1$, and suppose $\delta > 0$. Then $\exists n \in \mathbb{N}$, $n \geq 1$ s.t. $\frac{1}{n} < \delta$. Let $x_\delta = 2 + \frac{1}{n}$. Then $|x_\delta - 2| = \frac{1}{n} < \delta$, but $f(x_\delta) = \frac{1}{x_\delta - 2} = n$ satisfies

$$|f(x_\delta) - f(0)| = |f(x_\delta)| = n \geq \varepsilon.$$

i.e. $\forall \delta > 0$, $\exists x_\delta \in B(2, \delta)$ with $|f(x_\delta) - f(2)| \geq \varepsilon$.

Hence, f is <u>not</u> cts at $x = 2$.

2. Let A be a non-empty subset of $\mathbf{R}^n$ and consider the vector space of real-valued bounded functions

$$\mathcal{B}(A, \mathbf{R}) := \{f : A \to \mathbf{R} \mid f \text{ is bounded on } A.\}$$

Define a map $|.| : \mathcal{B}(A, \mathbf{R}) \to \mathbf{R}$ by

$$|f| := \sup\{|f(x)| \mid x \in A\}, \quad \forall f \in \mathcal{B}(A, \mathbf{R}).$$

a) Prove that for $f \in \mathcal{B}(A, \mathbf{R})$, then $|f| = 0 \iff f = 0$.

b) Prove that if $f \in \mathcal{B}(A, \mathbf{R})$, and $c \in \mathbf{R}$, then $|cf| = |c||f|$, where $|c|$, as usual, denotes the absolute value of the real number c.

c) Prove that $\forall f, g \in \mathcal{B}(A, \mathbf{R})$, $|f + g| \leq |f| + |g|$.

a) If $f = 0$, $\{|f(x)| \mid x \in A\} = \{0\}$, so $|f| = 0$. On the other hand, if $|f| = 0$, then, $\forall x \in A$, $0 \leq |f(x)| \leq |f| = 0$, so, $\forall x \in A$, $f(x) = 0$. Hence $f = 0$.

b) By (a), we may assume $c \neq 0$. First note that, $\forall x \in A$, $|c f(x)| = |c||f(x)| \leq |c||f|$ as $|f|$ is an upper bound for $B = \{|f(x)| \mid x \in A\}$. Now let $\varepsilon > 0$ be given, and choose $a \in A$ s.t. $|f| - \frac{\varepsilon}{|c|} < |f(a)| \leq |f|$. Then, $|c||f| - \varepsilon < |c f(a)| \leq |c||f|$

Thus, $|c||f|$ satisfies the properties of $\sup\{|c f(x)| \mid x \in A\}$, i.e. $|c||f| = |cf|$.

c) By def^n of $|f|, |g|$, $\forall x \in A$ $|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq |f| + |g|$

Hence $|f| + |g|$ is an upper bound for $C = \{|f(x) + g(x)| \mid x \in A\}$. Since $|f+g|$ is the least upper bound for C, $|f+g| \leq |f| + |g|$! Why? Suppose

$|f+g| > |f| + |g|$. Then, let $\varepsilon = |f+g| - |f| - |g|$, ; $\varepsilon > 0$ by assumption. For this ε, choose $a \in A$ s.t. $|f+g| - \varepsilon < |f(a) + g(a)| \leq |f+g|$. But then,

$$|f+g| - \varepsilon < |f(a) + g(a)| \leq |f(a)| + |g(a)| \leq |f| + |g|, \text{ i.e.}$$

$$|f+g| - \varepsilon < |f| + |g|. \qquad \text{But} \quad |f+g| - \varepsilon = |f| + |g|!$$

Hence $|f+g| > |f| + |g|$ is impossible.

<u>Remark</u> It is possible that $|f+g| < |f| + |g|$: e.g. let $A = [0,1]$, $f(x) = x$, $g(x) = -x$. Then $|f+g| = 0 < |f| + |g| = 2$.

3. Recall that $U \subseteq \mathbf{R}^n$ is *open* if $\forall x \in U$, $\exists r > 0$ such that $B(x,r) \subseteq U$.

a) Suppose $a \in \mathbf{R}^n$. Prove that for all $\varepsilon > 0$, $B(a,\varepsilon)$ is open.

b) Show that if $U_1, \ldots, U_n$ are open in $\mathbf{R}^n$, so too is $U_1 \cap \cdots \cap U_n$.

c) Suppose U is open in $\mathbf{R}^n$, $V = \mathbf{R}^n \setminus U$, and $\{v_m\}_{m\geq 1}$ is a sequence in V, i.e. $\{v_m \mid m \geq 1\} \subseteq V$. Prove that if $v \in \mathbf{R}^n$ is an accumulation point of $\{v_m\}_{m\geq 1}$, then $v \in V$.

d) Give an example of an infinite set $V \subseteq \mathbf{R}^n$, and a sequence $\{v_m\}_{m\geq 1}$ in V which has an accumulation point $v \notin V$.

a) 

Let $x \in B(a,\varepsilon)$, and set $r = \varepsilon - |x-a|$. As $|x-a| < \varepsilon$, $r > 0$.
We now show $B(x,r) \subseteq B(a,\varepsilon)$: Let $y \in B(x,r)$. So, $|x-y| < r$.
But $|a-y| = |a-x+x-y| \leq |a-x| + |x-y| < |a-x| + r = \varepsilon$.
i.e. $y \in B(x,r) \Rightarrow y \in B(a,\varepsilon)$. Hence, $B(x,r) \subseteq B(a,\varepsilon)$, and since $x \in B(a,\varepsilon)$ was arbitrary, $U = B(a,\varepsilon)$ is open.

b) Suppose $x \in U_1 \cap \ldots \cap U_n$, and each U_i is open, $1 \leq i \leq n$. For each i, choose $r_i > 0$ s.t. $B(x,r_i) \subseteq U_i$, $1 \leq i \leq n$. Let $r = \min(r_1, \ldots, r_n)$. Then $r > 0$, and $B(x,r) \subseteq B(x,r_i)$, $\forall i$, $1 \leq i \leq n$. Thus, $B(x,r) \subseteq \bigcap_{i=1}^{n} B(x,r_i) \subseteq \bigcap_{i=1}^{n} U_i$

Thus, $U_1 \cap \ldots \cap U_n$ is open.

c). Suppose not, i.e. $v \in \mathbb{R}^n \setminus V = U$. Since U is open, $\exists r > 0$ s.t. $B(v,r) \subseteq U$. Now choose any subsequence $\{v_{n_k}\}_{k\geq 1}$ with $\lim_{k\to\infty} v_{n_k} = v$.
For $\varepsilon = r > 0$, $\exists N$ s.t. $\forall k \geq N$, $|v - v_{n_k}| < \varepsilon = r$. That is, $\forall k \geq N$, $v_{n_k} \in B(v,\varepsilon) = B(v,r)$. But then, $\forall k \geq N$, $v_{n_k} \notin V$, since $B(v,r) \subseteq U = \mathbb{R}^n \setminus V$. This is a contradiction to the fact that $\{v_n\}_{n\geq 1} \subseteq V$. Hence, $v \notin \mathbb{R}^n \setminus V$, i.e. $v \in V$.

d) Let $v_m = (\frac{1}{m}, 0, \ldots, 0) \in \mathbb{R}^n$, $m \geq 1$, and $V = \{v_m \mid m \geq 1\}$. Then $0 \in \mathbb{R}^n$ is the limit of the sequence $\{v_m\}_{m\geq 1}$, i.e. $v_m \to 0$, but $0 \notin V$.
(Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$. Then $m \geq N \Rightarrow |v_m - 0| = |(\frac{1}{m}, 0, \ldots, 0)| = \frac{1}{m} < \varepsilon$.)

4. Suppose $g : [a, b] \to [c, d]$ is continuous at $x_0 \in [a, b]$, and $f : [c, d] \to \mathbf{R}$ is continuous at $g(x_0) \in [c, d]$. Prove carefully that the composite function $f \circ g : [a, b] \to \mathbf{R}$ defined by

$$(f \circ g)(x) := f(g(x)), \forall x \in [a, b],$$

is continuous at $x_0 \in [a, b]$.

Let $\varepsilon > 0$ and choose $\delta_0 > 0$ s.t. $|y - g(x_0)| < \delta_0 \Rightarrow |f(y) - f(g(x_0))| < \varepsilon$ (This is guaranteed by the continuity of f at $g(x_0)$.).

Then, for $\delta_0 > 0$, by the continuity of g at x_0, $\exists \delta_1 > 0$ s.t. $|x - x_0| < \delta_1 \Rightarrow |g(x) - g(x_0)| < \delta_0$. Hence, if $|x - x_0| < \delta_1$, $|f(g(x)) - f(g(x_0))| < \varepsilon$!