2120.A2 Solns
F08

1. Prove the following inequalities ($a, b, x, y, \varepsilon \in \mathbf{R}$):

a) $|x| \leq 1 \Rightarrow |x^2 - x - 2| \leq 3|x+1|$

b) $|x-a| < \varepsilon$ and $|y-b| < \varepsilon \Rightarrow |xy - ab| \leq (|a| + |b|)\varepsilon + \varepsilon^2$

a) $|x^2-x-2| = |(x-2)(x+1)| = |x-2||x+1|$. But, if $|x| \leq 1$,

$|x-2| \leq |x| + |-2| \leq 1+2 = 3$. Hence, $|x| < 1 \Rightarrow |x^2-x-2| < 3.|x+1|$.

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b) $|xy - ab| = |xy - bx + bx - ab| \leq |x||y-b| + |b||x-a|$.

We know $||x| - |a|| \leq |x-a|$, so in particular, $|x| - |a| \leq |x-a|$.

Hence $|x| \leq |x-a| + |a| < \varepsilon + |a|$, as $|x-a| < \varepsilon$. Thus

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$|xy - ab| \leq (\varepsilon + |a|)\varepsilon + |b|\varepsilon = (|a| + |b|)\varepsilon + \varepsilon^2$.

ie $|xy - ab| < (|a| + |b|)\varepsilon + \varepsilon^2$.

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(Rmk: $A < B \Rightarrow A \leq B$.)

2. Suppose $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ are Cauchy sequences of real numbers. Prove that

a) $\{a_n\}_{n\geq 1}$ is bounded (above and below).

b) $\{a_n b_n\}_{n\geq 1}$ is Cauchy.

idea that would wor

(1) some progress

+ (1) all is well

a) We know that every Cauchy sequence converges. We show (by mimicking the proof for rat'l sequence) that every convergent sequence of reals is bounded, and hence every Cauchy sequence is too. Suppose $\lim_{n\to\infty} a_n = a$. Let $\varepsilon = 1$, and choose $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a - a_n| < 1$. Then, $|a_n| \leq |a| + 1, \forall n \geq N$. (See this ass^n, Q1). Now let $K = \text{Max}\{|a_1|, \ldots, |a_{N-1}|, |a| + 1\}$. Hence, $|a_n| \leq K$, $\forall n \geq 1$, and so $\{a_n\}_{n\geq 1}$ is bdd.

b) We will do this directly: Let $m, n \in \mathbb{N}$. Then

as above.

$$|a_m b_m - a_n b_n| = |a_m b_m - a_m b_n + a_m b_n - a_n b_n|$$
$$\leq |a_m|\,|b_m - b_n| + |b_n|\,|a_m - a_n|. \quad *$$

Since $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ are Cauchy, by (a), we know both are bdd. Suppose $|a_n| \leq K$, and $|b_n| \leq L$, $\forall\, n \in \mathbb{N}$.

<u>Let $\varepsilon > 0$.</u> Since $\{a_n\}$ is Cauchy, $\exists N_a$ s.t. $\forall m, n \geq N_a$, $|a_m - a_n| < \frac{\varepsilon}{2L}$.

Similarly, as $\{b_n\}_{n\geq 1}$ is Cauchy, $\exists N_b$ s.t. $\forall m, n \geq N_a$, $|b_m - b_n| < \frac{\varepsilon}{2K}$.

Then, if $N = \max\{N_a, N_b\}$, and $n, m \geq N$, we have (by $*$)

$$|a_m b_m - a_n b_n| < K \cdot \frac{\varepsilon}{2K} + L \cdot \frac{\varepsilon}{2L} = \varepsilon$$

ie, $\forall n, m \geq N$, $|a_m b_m - a_n b_n| < \varepsilon$. Hence $\{a_n b_n\}_{n\geq 1}$ is Cauchy

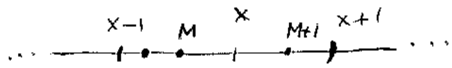
3. a) Suppose $x \in \mathbf{Q}$. Prove that $\exists N \in \mathbf{N}$ such that $N > x$.

b) Suppose $x \in \mathbf{R}$. (Hint: Let $\{x_n\}_{n\geq 1}$ be a sequence of rational numbers with $x_n \to x$, and use part (a).)

c) Suppose $a, b \in \mathbf{R}$ with $a < b$. Prove that there are integers m, n such that $a < \frac{m}{2^n} < b$.

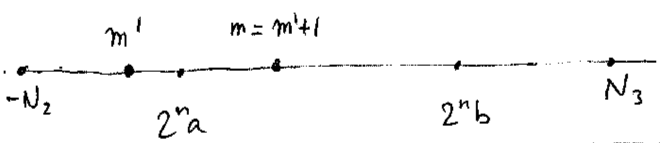
a) If $x < 0$, choose $N = 0$. If $x \geq 0$, choose $p, q \in \mathbb{N}$ with $q \neq 0$ s.t. $x = \frac{p}{q}$. We can write $p = s.q + r$ with $s \in \mathbb{N}$ and $0 \leq r < q$ (By the Euclidean Division Algorithm). Then, $\frac{p}{q} = s + \frac{r}{q}$, and $0 \leq \frac{r}{q} < 1$. Hence $s+1 > s + \frac{r}{q} = \frac{p}{q}$, and $s+1 \in \mathbb{N}$. (Set $N = s+1$). (1) idea ($\frac{1}{2}$) progress ($\frac{1}{2}$) well done

b) Let $\{x_n\}_{n \geq 1} \subset \mathbb{Q}$ be such that $x_n \to x$. With $\varepsilon = 1$, choose N_1 s.t. $n \geq N_1 \Rightarrow |x - x_n| < 1$. Then, $\forall n \geq N_1$ $x-1 < x_n < x+1$. In particular, $x - 1 < x_{N_1}$. Since $x_{N_1} \in \mathbb{Q}$, by part (a), $\exists M \in \mathbb{N}$ with $M > x_{N_1}$. Then, $M+1 > x_{N_1} + 1 > x$. Hence $M+1 > x$, and $M+1 \in \mathbb{N}$. (1) idea ($\frac{1}{2}$) progress ($\frac{1}{2}$) well done



c) By (b), choose $n \in \mathbb{N}$ s.t. $n > \frac{1}{(b-a)}$. Then $2^n > n > \frac{1}{(b-a)}$ (1) So $2^n(b-a) > 1$.
We show there is an integer in $(2^n a, 2^n b)$ (1) By (b) again, $\exists N_2 \in \mathbb{N}$ with $N_2 > -2^n a$, so $-N_2 < 2^n a$. Now let $S = \{k \in \mathbb{Z} \mid -N_2 \leq k \leq 2^n a\}$. Since (by (b) yet again) $\exists N_3 \in \mathbb{N}$ with $N_3 > 2^n a$, the set S is finite (it contains at most $N_2 + N_3$ elements.) Hence, $m' = \max S$** exists, and is an integer (since everything in S is.) Then, $m := m'+1$ satisfies $2^n a < m$ (since m' is the largest integer less than or equal to $2^n a$), and $m = m'+1 \leq 2^n a + 1 < 2^n b$. That is, $2^n a < m < 2^n b$, as req'd. Hence, $a < \frac{m}{2^n} < b$. + (1) a,b,c well written

(1) ← completing



If you used the existence of the supremum of a set bdd above in (a) or (b), your argument will be circular, since (our) proof of the existence of the sup used property (b) — look carefully at your notes.

** we could use sups here, but then we'd have to show $S \in \mathbb{Z}$.

4. a) Prove that $\{\sqrt{n} \mid n \in \mathbf{N}\}$ is unbounded (i.e. is not bounded). Indeed, prove *even more*, namely that $\forall K > 0$, $\exists N \in \mathbf{N}$ such that $\forall n \geq N, \sqrt{n} \geq K$.

b) Prove that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ exists, and find it. 1 idea

a) Let $K>0$ be given. By 3(b), $\exists$ N s.t. $N > K^2$. Since if $0<a,b$, then $0<a<b \iff a^2<b^2$; $\sqrt{N} > K$. Moreover, $n \geq N \Rightarrow \sqrt{n} \geq \sqrt{N}$ as well. Hence, $\forall n \geq N$, $\sqrt{n} \geq K$.

(We say in this case that "$\lim_{n\to\infty} \sqrt{n} = \infty$". 1 completion

b) We claim $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. Note first that $\sqrt{n+1} + \sqrt{n} \neq 0$,

Hence,
$$\sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

But, as shown in (a), $\sqrt{n+1} > \sqrt{n}$, so $\sqrt{n+1} + \sqrt{n} > 2\sqrt{n}$. Hence $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$. Now choose N s.t. $N > \frac{\varepsilon^2}{4}$. Then $\sqrt{N} > \frac{\varepsilon}{2}$ and $\frac{1}{2\sqrt{N}} < \varepsilon$. Hence, if $n \geq N$, $|\sqrt{n+1} - \sqrt{n}| < \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \varepsilon$.

Thus, $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

1 idea 1 completion

a, b well written

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5. a) Let $x, y \in \mathbf{R}$. Prove that $x \le y$ iff $\forall \varepsilon > 0, x \le y + \varepsilon$.

Now let $A, B \subset \mathbf{R}$ be non-empty. Define their sum by $A + B := \{a + b \mid a \in A, b \in B\}$.

b) Show that $A + B$ is bounded above iff both A and B are.

c) Prove that if $A+B$ is bounded above, then $\sup(A+B) = \sup A + \sup B$. (*Hint: First show that* $\sup(A+B) \le \sup A + \sup B$. *Then show that,* $\forall \varepsilon > 0$, $\sup A + \sup B \le \sup(A+B) + \varepsilon$, *and use part(a).*)

a) Clearly, $x \le y \Rightarrow (\forall \varepsilon > 0, x \le y + \varepsilon)$. So suppose $\forall \varepsilon > 0$, $x \le y+\varepsilon$, but that $x > y$. Let $\varepsilon = \frac{x-y}{2} (>0)$. Then, $x - y > \frac{x-y}{2} = \varepsilon$, ie, $x > y + \varepsilon$. This contradicts our assumption, so we must have $x \le y$. (Note that $(\forall \varepsilon > 0, x \le y + \varepsilon) \Leftrightarrow (\forall \varepsilon > 0, x < y + \varepsilon)$. The "$\Leftarrow$" part is obvious, so suppose $\forall \varepsilon > 0$, $x \le y + \varepsilon$. * If $\exists \varepsilon > 0$ with $x = y + \varepsilon$, set $\varepsilon' = \frac{\varepsilon}{2}$, and then we have $x > y + \varepsilon'$, contradicting *. Hence, $x < y + \varepsilon$, $\forall \varepsilon > 0$.)

b) Suppose $A+B$ is bdd above by K, and choose $a_0 \in A$, $b_0 \in B$. Then, $\forall a \in A$, $a + b_0 \le K$, so $\forall a \in A$, $a \le K - b_0$. Thus A is bdd above by $K - b_0$. Similarly, one can show that B is bdd above by $K - a_0$.

Now suppose A, B are bdd above by L_A and L_B respectively. Then, $\forall a+b \in A+B$, $a + b < L_A + L_B$, so $A+B$ is bdd above. (Remark*: the statement (b) is false if one of A or B is empty, but remains true if A and B are empty. Why?)

c)* Since $\forall a \in A$, $a \le \sup A$ and $\forall b \in B$, $b \le \sup B$, $\forall a+b \in A+B$, $a+b \le \sup A + \sup B$

Hence, $\sup A + \sup B$ is an upper bound for $A+B$.

Now let $\varepsilon > 0$. Then $\exists a \in A$ and $b \in B$ with $\sup A - \frac{\varepsilon}{2} < a$ and $\sup B - \frac{\varepsilon}{2} < b$. Hence, $\exists a+b \in A+B$ with $\sup A + \sup B - \varepsilon < a + b$ ($\le \sup(A+B)$, of course)

Thus, by defn, $\sup(A+B) = \sup A + \sup B$.

c)' (using the hint) We already showed that $\sup A + \sup B$ is an upper bnd for $A+B$, so $\sup(A+B) \le \sup A + \sup B$ (can you prove this?). The second part of the proof in c)* shows $\sup A + \sup B - \varepsilon < \sup(A+B)$, $\forall \varepsilon > 0$. By (a), this shows $\sup A + \sup B \le \sup(A+B)$, completing the proof.

* This proof doesn't use the hint.

* only for the very interested...

6. (Bonus) (See 1.15, P. 183) A sequence $\{s'_n\}_{n\geq 1}$ is called a *subsequence* of a sequence $\{s_n\}_{n\geq 1}$ if there is a function $\sigma : \mathbf{N} \to \mathbf{N}$ such that $n > m \Rightarrow \sigma(n) > \sigma(m)$, and $s'_n = s_{\sigma(n)}$, $\forall n \in \mathbf{N}$. (We often write a subsequence of $\{s_n\}_{n\geq 1}$ as $\{s_{n_k}\}_{k\geq 1}$, where n_k denotes $\sigma(k)$.)

(See 1.16, P. 183) A real number $s \in \mathbf{R}$ is called an *accumulation point* of a sequence $\{s_n\}_{n\geq 1}$ if there exists a subsequence (of $\{s_n\}_{n\geq 1}$) which converges to s.

(P. 186, 1.8)

a) Let $\{s_n\}_{n\geq 1}$ be a real bounded sequence. Prove that if $\{s_n \mid n \in \mathbf{N}\}$ has only one accumulation point, say, $s \in \mathbf{R}$, then the sequence converges and $\lim_{n\to\infty} s_n = s$.

b) Find a sequence $\{t_n\}_{n\geq 1}$ which is not bounded (and which therefore cannot converge) such that $\{t_n \mid n \in \mathbf{N}\}$ has only one accumulation pt.

a) Note that by Bolzano-Weierstrass, whenever a closed interval $[a,b]$ contains an infinite number of terms of any sequence, there will be a subsequence of the sequence which converges to a limit $L \in [a,b]$.

So suppose $\{s_n\}_{n\geq 1} \subset [x_0, y_0]$. Let $z_0 = \frac{x_0+y_0}{2}$ and consider the intervals $I_0 = [x_0, z_0]$, $J_0 = [z_0, y_0]$. If both contain infinitely many terms of $\{s_n\}_{n\geq 1}$, then there will be 2 subsequences, one in I_0, and one in J_0, with limit in I_0 and J_0. Unless these 2 limits are $z_0 \in I_0 \cap J_0$, we have a contradiction, since both limits are accumulation pts. In this case, $\forall n \geq 3$, $[a,b] - (z_0 - \frac{1}{n}, z_0 + \frac{1}{n})$ must have only finitely many terms of the sequence, otherwise (By B-W, one of the 2 intervals $[a, z_0 - \frac{1}{n}]$ or $[z_0 + \frac{1}{n}, b]$ would have an accumulation pt of $\{s_n\}_{n\geq 1}$ different from z_0, which is a contradiction. Hence, $\forall n$, $\exists N$ s.t. $\forall k \geq N$, $s_k \in (z_0 - \frac{1}{n}, z_0 + \frac{1}{n})$. Hence $\lim_{k\to\infty} s_k = z_0$. (If $L := [x, y]$ $(x<y)$, let $\mathring{L} := (x, y)$.)

So now suppose not both of I_0, J_0 contain infinitely many terms of the sequence, and let K_1 be the interval containing ∞ly many (so $[a,b] - \mathring{K}_1$ contains only finitely many terms.) Divide K_1 in 2 equal halves as before, and argue as before. Either $\lim_{k\to\infty} s_k$ is the midpoint of K_1 or, one of the halves of K, contains only finitely many terms of the sequence. Let K_2 be the other half. We continue in this way, either stopping if $\lim_{k\to\infty} s_k$ is the midpoint of some K_N, or we obtain a infinite nested sequence of intervals $K_{n+1} \subset K_n$ with length $K_n = \frac{1}{2^n}(b-a)$. By the nested interval lemma, $\bigcap_{n\geq 1} K_n = \{s\}$. Moreover, at each stage, $[a,b] - K_n$ contains only finitely many elements of the sequence. i.e. $\forall n$, $\exists N$ s.t. $\forall k \geq N$, $s_k \in K_n$. Thus, $\forall n$, $\exists N$ s.t. $|s_k - s| \leq \frac{b-a}{2^n}$. This shows $\lim_{k \to \infty} s_k = s$. Hence when $\{s_n\}_{n \geq 1}$ has only one accumulation pt, the sequence converges to it.

b) Let $t_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases}$.

Then $\{t_n\}_{n \geq 1}$ is not bdd, and has only one accumulation point, namely 0.