1. Prove carefully that

$$\forall x, y \in \mathbf{Q}, \quad |\,|x| - |y|\,| \leq |x - y|.$$

You may assume that the triangle inequality holds for $\mathbf{Q}$.

Let $a = y$ and $b = x - y$. Then, by the Δ ineq.

$$|a + b| \leq |a| + |b|, \text{ i.e.}$$

$$|x| \leq |y| + |x - y|.$$

Hence $$|x| - |y| \leq |x - y|.$$

Interchanging the roles of x and y in the above, we obtain

$$|y| - |x| \leq |y - x| = |x - y|.$$

Hence, $$|x - y| \geq \max(|x| - |y|, -(|x| - |y|))$$
$$= |\,|x| - |y|\,|.$$

2. Suppose $a, b \in \mathbf{Q}$, and that $\{a_n \mid n \geq 1\}$ and $\{b_n \mid n \geq 1\}$ are sequences of rational numbers such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Carefully prove that

(a) $\lim_{n\to\infty} 3a_n + 2 = 3a + 2$

(b) $\lim_{n\to\infty} 3a_n - 2b_n = 3a - 2b$

(a) Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ s.t. $n \geqslant N \Rightarrow |a - a_n| < \frac{\varepsilon}{3}$. Then,

$n \geqslant N \Rightarrow |(3a+2) - (3a_n+2)| = |3(a-a_n)| = 3|a-a_n| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

That is, $n \geqslant N \Rightarrow |(3a+2) - (3a_n+2)| < \varepsilon$. Hence,

$$\lim_{n\to\infty} 3a_n + 2 = 3a+2.$$

(b) Let $\varepsilon > 0$ and choose $N_a, N_b \in \mathbb{N}$ s.t. $n \geqslant N_a \Rightarrow |a - a_n| < \frac{\varepsilon}{6}$ and $n \geqslant N_b \Rightarrow |b - b_n| < \frac{\varepsilon}{4}$. Let $N = \max(N_a, N_b)$.

Then, $|(3a-2b) - (3a_n - 2b_n)| = |3(a-a_n) - 2(b-b_n)|$

$\leqslant |3(a-a_n)| + |2(b-b_n)| = 3|a-a_n| + 2|b-b_n|$.

Hence, $n \geqslant N \Rightarrow |(3a-2b) - (3a_n - 2b_n)| < 3 \cdot \frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon$.

i.e. $n \geqslant N \Rightarrow |(3a-2b) - (3a_n - 2b_n)| < \varepsilon$.

Hence $\lim_{n\to\infty} 3a_n - 2b_n = 3a - 2b$

3. (Text, P 185, #1.2) Define a sequence $\{s_n \mid n \geq 1\}$ by

$$s_n = \frac{2n-1}{n+3}.$$

(a) Prove carefully that $\lim_{n \to \infty} s_n = 2$.

(b) If $\varepsilon = 10^{-5}$, find an integer N such that

$$n \geq N \Rightarrow |s_n - 2| < \varepsilon.$$

(a) Note that $|2 - s_n| = \left| \frac{2n+6-2n+1}{n+3} \right| = \frac{7}{n+3}.$

Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ st. $N > \frac{7}{\varepsilon} - 3$.

Hence, $N + 3 > \frac{7}{\varepsilon}$, so $\varepsilon > \frac{7}{N+3}$.

Then, $n \geqslant N \Rightarrow |2 - s_n| = \frac{7}{n+3} \leqslant \frac{7}{N+3} < \varepsilon$.

Thus, $\lim_{n \to \infty} s_n = 2$.

(b) If $\varepsilon = 10^{-5}$, $\frac{7}{\varepsilon} - 3 = 7 \times 10^5 - 3$.

Hence, let $N = 7 \times 10^5 - 2$. Then,

$$n \geqslant N \Rightarrow |2 - s_n| < 10^{-5}.$$

4. (Text, P 185, #1.3, first part only.) Define a sequence $\{s_n \mid n \geq 1\}$ by

$$s_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)}.$$

(a) Find a partial fraction decomposition of $\frac{1}{(2k-1)(2k+3)}$.

(b) Prove carefully that $\{s_n \mid n \geq 1\}$ is a Cauchy sequence. (c) Find $\lim\limits_{n\to\infty} s_n$, and prove it exists.

(a) A quick calculation shows that $\frac{1}{(2k-1)(2k+3)} = \frac{1}{4}\left(\frac{1}{2k-1} - \frac{1}{2k+3}\right).$

(b) & (c)

Hence, $S_n = \frac{1}{4}\Big[\left(1 - \cancel{\tfrac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{9}}\right) + \left(\cancel{\tfrac{1}{7}} - \frac{1}{11}\right) + \left(\cancel{\tfrac{1}{9}} - \cancel{\tfrac{1}{13}}\right) + \cdots$

$+ \cdots + \left(\cancel{\tfrac{1}{2n-7}} - \cancel{\tfrac{1}{2n-3}}\right) + \left(\cancel{\tfrac{1}{2n-5}} - \cancel{\tfrac{1}{2n-1}}\right) + \left(\cancel{\tfrac{1}{2n-3}} - \frac{1}{2n+1}\right) + \left(\cancel{\tfrac{1}{2n-1}} - \frac{1}{2n+3}\right)\Big]$

$$= \frac{1}{4}\left[1 + \frac{1}{3} - \frac{1}{2n+1} - \frac{1}{2n+3}\right]. = \frac{1}{3} - \frac{1}{4}\left[\frac{1}{2n+1} + \frac{1}{2n+3}\right]$$

To show $\{s_n\}_{n\geq 1}$ is Cauchy, we show the limit exists. We prove $\lim\limits_{n\to\infty} s_n = \frac{1}{3}$ as follows: $\left|\frac{1}{3} - s_n\right| = \left|\frac{1}{4}\left(\frac{1}{2n+1} + \frac{1}{2n+3}\right)\right|$

$\leq \frac{1}{4}\cdot\left(\frac{1}{2n+1} + \frac{1}{2n+3}\right) < \frac{1}{4}\cdot\frac{1}{2n} + \frac{1}{2n} = \frac{1}{16n}.$ Let $\varepsilon > 0$

and choose $N \geq \frac{1}{16\varepsilon}$. Then $n \geq N \Rightarrow n \geq \frac{1}{16\varepsilon}$ so $\varepsilon \geq \frac{1}{16n}$;

moreover, $n \geq N \Rightarrow \left|\frac{1}{3} - s_n\right| < \frac{1}{16n} \leq \varepsilon$; ie $n \geq N \Rightarrow \left|\frac{1}{3} - s_n\right| < \varepsilon$.

Hence $\lim\limits_{n\to\infty} s_n = \frac{1}{3}$;

By a theorem in class, every convergent sequence is Cauchy. (this could also be proven directly, using the expression from (a), and estimating as in the second part.)

5. (Bonus) Suppose a sequence $\{a_n \mid n \geq 1\}$ of positive rational numbers satisfies $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$.

Prove carefully that $\lim_{n\to\infty} s_n = 0$.

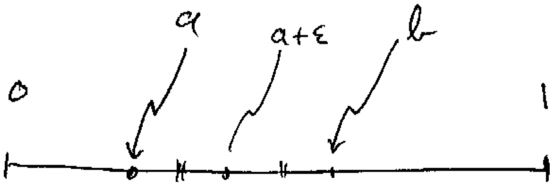
(*Hint: Proceed as follows.*

a) Prove that if $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$, *then* $\exists N \in \mathbf{N}$ *and* $s > 0$ *s.t* $n \geq N \Rightarrow 0 < \frac{s_{n+1}}{s_n} < \frac{1}{1+s}$.

b) Prove by induction that if $s > 0$, *then* $(1+s)^k > ks$, *for* $k \geq 1$.

c) Prove that if $s > 0$, $\lim_{k\to\infty} \frac{1}{(1+s)^k} = 0$.

d) Carry on!)



a) Let $a = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, $b = \frac{a+1}{2}$ and $\varepsilon = \frac{b-a}{2}$. Since $a<1$, we know $a<b$ and so $\varepsilon > 0$. Thus, $\exists N$ s.t. $\forall n \geq N$, we have $\left|a - \frac{s_{n+1}}{s_n}\right| < \varepsilon$. In particular, $\frac{s_{n+1}}{s_n} - a < \varepsilon$, so that $0 < \frac{s_{n+1}}{s_m} < a+\varepsilon < b$. Define $s = \frac{1}{b} - 1$, so that $b = \frac{1}{1+s}$. As $b<1$, $s>0$. Hence, $n \geq N \Rightarrow 0 < \frac{s_{n+1}}{s_n} < \frac{1}{1+s}$, and $s>0$.

b) We know from a DUD that $(1+s)^k \geq 1+ks > ks$, for all $k \geq 1$. (One could prove $(1+s)^k > ks$ by induction, but its even then easier to prove more: i.e. that $(1+s)^k \geq 1+ks$.)

c) We know $\frac{1}{(1+s)^k} < \frac{1}{ks}$, so let $\varepsilon > 0$, and choose $N > \frac{1}{s\varepsilon}$. Then, $k \geq N \Rightarrow k > \frac{1}{s\varepsilon}$, or $\frac{1}{ks} < \varepsilon$. Hence, $\forall k \geq N$, $\left|0 - \frac{1}{(1+s)^k}\right| < \frac{1}{ks} < \varepsilon$.

i.e. $\lim_{k\to\infty} \frac{1}{(1+s)^k} = 0$.

d) Let N_0 be as in part (a). If $m \geq N_0$, $s_m = \frac{s_m}{s_{m-1}} \cdot \frac{s_{m-1}}{s_{m-2}} \cdots \frac{s_{N_0+1}}{s_{N_0}} \cdot s_{N_0}$

$$< \frac{s_{N_0}}{(1+s)^{m-N_0}}$$, by (a).

5(d) cont. Now let $\varepsilon > 0$ and choose N_1 s.t. $\frac{1}{(1+\delta)^k} < \frac{\varepsilon}{s_{N_0}}$ if $k \geq N_1$. Now let $N_2 = \max(N_0, N_1 + N_0)$. $(= N_1 + N_0)$.

Then $m \geq N_2 \Rightarrow m - N_0 \geq N_1$, so

$$|0 - s_m| = s_m < s_{N_0} \cdot \frac{\varepsilon}{s_{N_0}} = \varepsilon.$$ Hence,

$$\lim_{m \to \infty} s_m = 0.$$