2120 F09 Final Solns

1. a) If $A \subset \mathbf{R}$ give the definition of the *supremum* of A, usually denoted "sup A". 2

b) Let $\{a_n\}_{n\geq 1}$ be a bounded real sequence and let

$$a := \sup\{a_n \mid n \geq 1\}.$$

idea 1
S_k infinite 1
$n_{k+1} > n_k$ 3

Suppose, in addition, that

$$\forall n \geq 1, \quad a_n < a.$$

Prove that a is an accumulation point of $\{a_n\}_{n\geq 1}$.

c) Given an example of a bounded sequence $\{b_n\}_{n\geq 1}$ for which $\sup\{b_n\}_{n\geq 1}$ is *not* an accumulation point of $\{b_n\}_{n\geq 1}$. 2

1 - correct
1 - pf [illegible] acc pt

a) $x = \sup A \Leftrightarrow$ (1) $\forall a \in A, \quad a \leq x$ & (2) $\forall \varepsilon > 0, \exists a \in A$ s.t. $x - \varepsilon < a \leq x$

b) Let $k \geq 1$, $k \in \mathbb{N}$ and define $S_k = \{n \in \mathbb{N} \mid a_n \in (a - \frac{1}{k}, a)\}$. If S_k were finite for any k, it would have a maximal element, say $s = a_{n'}$. Then, $\forall n$, $a_n \leq a_{n'}$, and since $a_{n'} < a$, $a_{n'}$ would be an upper bound which is strictly less than a, contradicting the definition of a.

Since each S_k is infinite, we can choose $n_k \in S_k$, $\forall k \geq 1$, such that $n_{k+1} > n_k$. Then $\{a_{n_k}\}_{k\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$

and since $|a - a_{n_k}| < \frac{1}{k}$, $\forall k \geq 1$, $a_{n_k} \to a$. Hence a is an accumulation point of $\{a_n\}_{n\geq 1}$.

c) Let $b_n = \frac{1}{n}$, $n \geq 1$. Then $\sup\{b_n\}_{n\geq 1} = b_1 = 1$, but since $\lim_{n\to\infty} b_n = 0$, 0 is the only accumulation point of $\{b_n\}_{n\geq 1}$. (Every subsequence of a convergent subsequence converges to the same limit as the whole sequence.)

2. a) State your favourite version of the Mean Value Theorem. (2) See your notes or the text.

b) Prove that if a, c and $x \in \mathbf{R}$,

$$|x| > c > 0 \implies \left|\frac{1}{a} - \frac{1}{x}\right| \leq \frac{|x-a|}{c|a|}.$$ (1)

c) Use (b) and the definition of continuity to prove that $f(x) = \dfrac{1}{x}$ is continuous on $\mathbf{R} \setminus \{0\}$. (2)

Now let $g : \mathbf{R} \to \mathbf{R}$ be differentiable everywhere. Suppose further that g' is non-negative and decreasing on $[0, \infty)$. That is,

(I) $g'(x) \geq 0, \forall x \in [0, \infty)$, and

(II) $x \leq y \implies g'(x) \geq g'(y) \quad \forall x, y \in [0, \infty)$.

d) Prove that the function g' is bounded on $[0, \infty)$. (1)

e) Use (d) to conclude that there exists M such that (1)

$$|g(x) - g(y)| \leq M|x - y|, \quad \forall x, y \in [0, \infty).$$

(a) –

(b) $\left|\frac{1}{a} - \frac{1}{x}\right| = \dfrac{|x-a|}{|a||x|} < \dfrac{|x-a|}{|a|c}$, if $|x| > c > 0$

(c) Let $a \in \mathbb{R}\setminus\{0\}$ and $\varepsilon > 0$. Then $|a| > 0$. Suppose $|x-a| < \frac{|a|}{2}$. Then, $|a| = |a - x + x| \leq |a-x| + |x| < \frac{|a|}{2} + |x|$, and so $|x| > \frac{|a|}{2}$. Hence, if $|x-a| < \frac{|a|}{2}$, by (a) $\left|\frac{1}{a} - \frac{1}{x}\right| \leq 2\dfrac{|x-a|}{|a|^2}$. Now let $\delta = \min\left(\frac{|a|}{2}, \frac{\varepsilon|a|^2}{2}\right)$. Then $|x-a| < \delta \Rightarrow \left|\frac{1}{a} - \frac{1}{x}\right| < \frac{2}{|a|^2}\cdot\frac{\varepsilon|a|^2}{2} = \varepsilon$. Hence, f is cts at $a \in \mathbb{R}\setminus\{0\}$. Since a was arbitrary, f is cts on $\mathbb{R}\setminus\{0\}$.

(d) $\forall x \in [0, \infty)$, $g'(x) \leq g'(0)$, so g' is bounded on $[0, \infty)$.

(e) By the (Lagrange) MVT applied to g on $[x, y]$, $\exists \xi \in (x, y)$ s.t. $\dfrac{g(x) - g(y)}{x - y} = g'(\xi)$. Hence, $|g(x) - g(y)| = |g'(\xi)||x-y| \leq |g'(0)||x-y|$, since $\xi \in (0, \infty)$.

3. a) If $K \subset \mathbf{R}^p$, give two different characterizations of "K is compact". (One of them may be the definition.) (2)

b) Let A and B be two compact subsets of $\mathbf{R}^2$. Using either of the characterizations of compactness in (a), prove that $A \cap B$ is also compact. (3)

c) Let $C = \{v \in \mathbf{R}^2 \mid 1 \leq \|v\| \leq 2\}$. Using any method, prove that C is compact. (2)

a) See your notes or the text.

b) If A and B are compact, they are both bounded. Since any subset of a bounded set is bounded, and $A \cap B \subseteq A$, $A \cap B$ is bounded. Now suppose $\{v_n\}_{n \geq 1} \subseteq A \cap B$ and $v_n \to v \in \mathbb{R}^2$. Since A is closed, and $\{v_n\}_{n \geq 1} \subseteq A$, $v \in A$. Similarly, $v \in B$. Hence, $v \in A \cap B$ and so $A \cap B$ is closed.

Hence, $A \cap B$ is closed & bdd and so is compact.

c) Since $\left| \|v\| - \|w\| \right| \leq \|v - w\|$, $\forall v, w \in \mathbb{R}^2$, $g: \mathbb{R}^2 \to \mathbb{R}$ def'd by $g(v) = \|v\|$ is cts ("$\delta = \varepsilon$"). Moreover, we know $[1, 2] \subset \mathbb{R}$ is closed, Hence, $g^{-1}([1, 2]) = C$ is also closed, as it is the preimage of a closed set by a cts function.

Since $\|v\| \leq 2$, $\forall v \in C$, C is bounded.

Hence, C is compact.

4. a) Suppose $f_n : [a,b] \to \mathbf{R}, n \geq 1$ is a sequence of functions. Define what is meant by

$$\sum_{n=1}^{\infty} f_n(x) \quad \textit{converges uniformly on } [a,b].$$ 2

b) (i) Give the Taylor polynomial of order 3, with remainder, at 0, for the function

$$g(x) = \frac{1}{6-x}.$$ 1 + 1

(ii) Show that $g^{(n)}(0) = \frac{n!}{6^{n+1}}$. 1/2

(iii) Prove that the Taylor series for g at 0 converges uniformly on any closed interval $[-r, r]$, $r < 6$. 2

c) If the Taylor series of a function $f : \mathbf{R} \to \mathbf{R}$ at 0 converges at a point $x \in \mathbf{R}$, must it converge to $f(x)$? (No justification is necessary.) 1/2

a) $\sum_{n\geq 1} f_n(x)$ converges uniformly on $[a,b]$ if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, p \geq N$ and $\forall x \in [a,b]$ $\left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{p} f_k(x) \right| < \varepsilon$

b) We first show $g^{(n)}(x) = \frac{n!}{(6-x)^{n+1}}$. For $n=0$, this is clear, so suppose this is true for $n \in \mathbb{N}$. Then $g^{(n+1)}(x) = g^{(n)\prime}(x) = \frac{(-1)-(n+1)n!}{(6-x)^{n+2}} = \frac{(n+1)!}{(6-x)^{n+2}}$.

Hence by induction $g^{n}(x) = \frac{n!}{(6-x)^{n+1}}$, $\forall n \geq 0$. This establishes (ii).

By Taylor's theorem, $g(x) = g(0) + g'(0)x + g''(0)\frac{x^2}{2} + g^{(3)}(0)\frac{x^3}{3!} + g^{(4)}(\xi)\frac{x^4}{4!}$ for some ξ. So $P_3(x) = \frac{1}{6} + \frac{x}{6^2} + \frac{x^2}{6^3} + \frac{x^3}{6^4}$ and the remainder $R_3(x) = \frac{x^4}{(6-\xi)^5}$ for some ξ between 0 and x.

(b)(iii) The Taylor series for g at 0 is $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}}$

$= \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{x}{6}\right)^n$. If $|x| \leq r < 6$, then $\left|\frac{x}{6}\right|^n \leq \left(\frac{r}{6}\right)^n$, and $\sum \left(\frac{r}{6}\right)^n$ converges because $\left|\frac{r}{6}\right| < 1$. Hence, by the Weierstrass M-Test, $\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{6}\right)^n$ converges uniformly on $[-r, r]$. Hence, the Taylor series for g at 0 converges uniformly on $[-r, r]$ if $0 \le r < 6$.

[Note: Since $\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{6}\right)^n = \frac{1}{6}\left(\frac{1}{1-\frac{x}{6}}\right) = \frac{1}{6-x}$ for
(geometric series!)

in fact $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, x^n}{n!}$ for $x \in (6, 6)$.]

c) No, e.g. let $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then we saw in class that the Taylor series for f at 0 is identically zero, and f is not.

5. Suppose $U \subseteq \mathbf{R}^2$ is open, and $a \in U$.

a) Suppose $g : U \to \mathbf{R}^2$. Define "g is continuous at a." 1

b) Suppose $f : U \to \mathbf{R}$. Define "f is differentiable at a." 2

c) Now suppose $g : \mathbf{R}^2 \to \mathbf{R}^2$ is continuous at $0 \in \mathbf{R}^2$, and define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(v) = g(v) \cdot v, \quad \forall v \in \mathbf{R}^2 \qquad \text{(dot product)}$$

Prove that f is differentiable at $0 \in \mathbf{R}^2$, and that 4

$$df(0)(v) = g(0) \cdot v, \quad \forall v \in \mathbf{R}^2 \qquad \text{(dot product)}$$

(*Hint: You may use the inequality*

$$\left|\Big(g(v) - g(0)\Big) \cdot v\right| \leq \|g(v) - g(0)\|\|v\|, \quad \forall v \in \mathbf{R}^2.\ )$$

a) See your note b) See your note

Note that $v \mapsto g(0) \cdot v$ is a linear map $\mathbb{R}^2 \to \mathbb{R}$.

Moreover, $\dfrac{f(v) - f(0) - g(0)\cdot(v-0)}{\|v-0\|} = \underbrace{\dfrac{g(v)\cdot v - g(0)\cdot v}{\|v\|}}_{2}$

$= \dfrac{\big(g(v) - g(0)\big)\cdot v}{\|v\|}$. Thus $\dfrac{|f(v) - f(0) - g(0)\cdot(v-0)|}{\|v-0\|}$

$= \dfrac{|\big(g(v)-g(0)\big)\cdot v|}{\|v\|} \leq \dfrac{\|g(v) - g(0)\|\,\|v\|}{\|v\|} = \|g(v) - g(0)\|$. 1

Since g is cts at 0, $\lim_{v \to 0} \|g(v) - g(0)\| = 0$, and so 1

$\lim_{v \to 0} \dfrac{f(v) - f(0) - g(0)\cdot(v-0)}{\|v\|} = 0$. Hence

$df(0)$ exist, and $df(0)(v) = g(0)\cdot v$

6. Define a function $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \sqrt{|xy|}.$$

2 a) Prove that the function is continuous at $(0, 0)$,

1 b) Compute the partial derivatives at $(0, 0)$, if possible. 1 + 1

2 c) Decide whether or not the function is differentiable at $(0, 0)$. 2

1 d) Are the partial derivatives of f continuous at $(0, 0)$? 1 "reason"

a) Since $|x|$ and $|y|$ are bdd above by $\|(x,y)\|$, we have

$$|f(x,y) - f(0,0)| = \sqrt{|xy|} \leq \sqrt{\|(x,y)\|\,\|(x,y)\|} = \|(x,y)\|.$$ Hence,

f is cts at $(0,0)$ (Choose "$\delta = \varepsilon$".)

b) $$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0.$$ Similarly,

$$\frac{\partial f}{\partial y}(0,0) = 0.$$

c) Consider $$\frac{f(v) - f(0) - [0\ 0](v-0)}{\|v\|} = \frac{\sqrt{|xy|}}{\|(x,y)\|} = \frac{\sqrt{|x||y|}}{\sqrt{x^2+y^2}}, \text{ if } v=(x,y)$$ 1

If $v = (t,0)$ $(=(x,y))$, then $$\lim_{t\to 0} \frac{\sqrt{|x||y|}}{\sqrt{x^2+y^2}} = 0,$$ while if

$v = (t,t)$, $$\lim_{t\to 0}\frac{\sqrt{|x||y|}}{\sqrt{x^2+y^2}} = \lim_{t\to 0} \frac{|t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}} \neq 0.$$

Hence $$\lim_{v\to 0} \frac{f(v) - f(0) - \left[\frac{\partial f}{\partial x}(0,0)\ \ \frac{\partial f}{\partial y}(0,0)\right](v-0)}{\|v-0\|}$$ does not exist. 1

Thus $f'(0,0)$ does not exist.

d) No, since if f were C^1 at $(0,0)$, $f'(0,0)$ would exist.

7. Let $f(x,y) = x^2 + y^2 + 3$, and define $h : \mathbf{R}^2 \to \mathbf{R}^3$ by

$$h(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}.$$

[2] a) Find the Jacobian matrix of f at $(1,-1)$, and the Jacobian matrix of h at $(1,-1)$. 1 1

[2] b) Find ~~the~~ an equation of the tangent plane to the graph of f, $G_f = \{(x,y,f(x,y)) \mid (x,y) \in \mathbf{R}^2\}$ at the point $(1,-1)$. 2

[1] c) Find the direction at $(1,-1)$ in which f *increases* the fastest. 1

d) Use Lagrange multipliers to find the maximum and minimum values of f on the compact set

$$\{(x,y) \mid x^4 + y^4 \le 2\}.$$ 2

a) $df(1,-1) = \left[\frac{\partial f}{\partial x}(1,-1) \quad \frac{\partial f}{\partial y}(1,-1)\right] = [2 \quad -2]$, since $\frac{\partial f}{\partial x} = 2x$ & $\frac{\partial f}{\partial y} = 2y$

$h' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$ at $(1,-1)$, by the previous computation.

b) The normal to G_f is $\nabla k(1,-1,5)$, where $k(x,y,z) = z - f(x,y)$, as $G_f = \{v \in \mathbb{R}^3 \mid k(v) = 0\}$ and k is differentiable at $(1,-1,5)$.

As $\nabla k(1,-1,5) = \begin{bmatrix} -\frac{\partial f}{\partial x}(1,-1) \\ -\frac{\partial f}{\partial y}(1,-1) \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, an equation to the tangent plane to G_f at $(1,-1)$ is $-2(x-1) + 2(y+1) + z - 5 = 0$, or

$-2x + 2y + z = 1$

c) This is $\dfrac{\nabla f(1,-1)}{\|\nabla f(1,-1)\|} = \dfrac{\begin{bmatrix} 2 \\ -2 \end{bmatrix}}{2\sqrt{2}} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$.

d) $\nabla f(x,y) = 0 \iff x=y=0$, and $f(0,0)=3$. 1 $(0^2+0^2<2)$.

Moreover, if we set $g(x,y) = x^4+y^4-2$ then

$g(x,y)=0$ and $\nabla f(x,y) = \lambda \nabla g(x,y) \iff$

$x^4+y^4=2$ and $\begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}$. $\iff x^4+y^4=2$ and

$0 = x(1-2\lambda x^2)$ & , If $x=0$, $y=\pm\sqrt[4]{2}$ and $f(0,\pm\sqrt[4]{2}) = \sqrt{2}+3$
$0 = y(1-2\lambda y^2)$ $\quad y=0$, $x=\pm\sqrt[4]{2}$ and $f(\pm\sqrt[4]{2},0) = \sqrt{2}+3$

If $x\neq 0$ and $y\neq 0$, then $1 = 2\lambda x^2 = 2\lambda y^2$. Since $\lambda=0$ is impossible, $x^2=y^2$ and so $2x^4=2y^4=2$. Hence $(x,y) = (\pm 1, \pm 1)$. Since $f(\pm1,\pm1) = 5 > \sqrt{2}+3 > 3$ (since $2>\sqrt{2}$), the maximum value of f is $\underline{5}$, and the minimum value of f is $\underline{3}$ (on the indicated compact set.)

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8. (Bonus) Let K be a compact set in $\mathbf{R}^n$. If an *increasing* sequence $\{f_n\}_{n\geq 1}$ of continous real-valued functions $f_n : K \to \mathbf{R}$ converges to a continuous function $f : K \to \mathbf{R}$, prove that $f_n \to f$ *uniformly* on K. (5)

Since K is compact and f & f_n are cts $\forall n \in \mathbb{N}$, $f - f_n$ is cts on K and so $M_n = \max\{f(x) - f_n(x) \mid x \in K\}$ exists and is attained at, say $x_n \in K$. As $\{f_n\}_{n\geq 1}$ is increasing, $\{f - f_n\}$ is decreasing, and so $M_{n+1} \leq M_n$, $\forall n \in \mathbb{N}$. Moreover, since $f(x) = \sup\{f_n(x) \mid n \in \mathbb{N}\} = \lim_{n\to\infty} f_n(x)$, $\forall x \in K$, $f - f_n \geq 0$. Hence $\{M_n\}_{n\geq 1}$ is a decreasing sequence bdd below by 0. Hence $M = \lim_{n\to\infty} M_n$ exists. It suffices to show that $M = 0$.

Suppose not, i.e. $M > 0$. Then as K is compact and $\{x_n\}_{n\geq 1} \subseteq K$, there is a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ s.t. $x_{n_k} \to a \in K$.

Set $g_n = f - f_n$. Then g_n is cts $\forall n$, $\{g_n\}_{n\geq 1}$ is decreasing, positive, & $g_n \to 0$ pointwise on K. Note that since $M > 0$ and $g_n(a) \to 0$, $\exists K \in \mathbb{N}$ s.t., $\forall k \geq K$, $g_{n_k}(a) < M/2$.

Moreover, since g_{n_K} is cts at a, $\exists L \in \mathbb{N}$ s.t. $\forall l \geq L$, $|g_{n_K}(a) - g_{n_K}(x_{n_l})| < \frac{M}{2}$. If $K \geq L$,

$$M \leq M_{n_K} = g_{n_K}(x_{n_K}) \leq |g_{n_K}(x_{n_K}) - g_{n_K}(a)| + |g_{n_K}(a)| < M,$$

a contradiction. If $K < L$,

$$M \leq M_{n_L} = g_{n_L}(x_{n_L}) \leq g_{n_K}(x_{n_L}) \leq |g_{n_K}(x_{n_L}) - g_{n_K}(a)| + |g_{n_K}(a)|$$
$$< \frac{M}{2} + \frac{M}{2} = M,$$ another contradiction. Hence $M = 0$, and $g_n \to 0$ uniformly on K, i.e. $f_n \to f$ uniformly on K.