

This examination has 14 pages including this cover

The University of British Columbia

Final Examination – 12 June 2009

Mathematics 220

Mathematical Proof

Closed book examination

Time: 150 minutes

Name _____ Signature _____

UBC Student Number _____

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed.

Rules governing examinations

1. All candidates should be prepared to produce their library/AMS cards upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		5
2		5
3		8
4		8
5		8
6		10
7		9
8		9
9		9
10		10
11		10
12		9
Total		100

- [5] 1. Prove: For logical statements P , Q , and R , $[(P \wedge Q) \Rightarrow R] \equiv P \Rightarrow (Q \Rightarrow R)$.

$$\begin{aligned}
 (P \wedge Q) \Rightarrow R &\equiv R \vee \sim(P \wedge Q) && \text{def}^n \text{ of } "\Rightarrow" \\
 &\equiv R \vee (\sim P) \vee (\sim Q) && \text{de Morgan} \\
 &\equiv (R \vee \sim Q) \vee (\sim P) && \text{associative law} \\
 &\equiv (Q \Rightarrow R) \vee (\sim P) && \text{def}^n \text{ of } "\Rightarrow" \\
 &\equiv P \Rightarrow (Q \Rightarrow R) && \text{def}^n \text{ of } "\Rightarrow"
 \end{aligned}$$

- [5] 2. The distributive law is a famous fact valid for any sets A , B , and C :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Prove this equation from first principles.

(\subseteq) Pick any object x .

$$\begin{aligned}
 x \in A \cap (B \cup C) &\Rightarrow x \in A \wedge x \in (B \cup C) \\
 &\Rightarrow (x \in A) \wedge [(x \in B) \vee (x \in C)] \\
 &\Rightarrow [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\
 &\Rightarrow [x \in A \cap B] \vee [x \in A \cap C] \\
 &\Rightarrow x \in (A \cap B) \cup (A \cap C). \quad \text{Done.}
 \end{aligned}$$

(\supseteq) Pick any object x . Every implication above actually works also in reverse. So if $x \in \text{RHS}$, then $x \in \text{LHS}$. Done.

[8] 3. Let A and B be nonempty sets, and suppose $f: A \rightarrow B$. Prove: If $A_1, A_2 \subseteq A$, then

- (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
 (b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
 (c) If f is one-to-one, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

(a) (\subseteq) Pick $y \in f(A_1 \cup A_2)$. This means $y = f(x)$ for some $x \in A_1 \cup A_2$.

If $x \in A_1$, then $y \in f(A_1) \subseteq f(A_1) \cup f(A_2)$; if $x \in A_2$, then $y \in f(A_2) \subseteq f(A_1) \cup f(A_2)$. In either case, $y \in \text{RHS}$.

(\supseteq) Pick $y \in f(A_1) \cup f(A_2)$. This means either $y \in f(A_1)$ or $y \in f(A_2)$ [or both]. If $y \in f(A_1)$ then $y = f(x)$ for some $x \in A_1 \subseteq A_1 \cup A_2$, so $y \in f(A_1 \cup A_2)$. It's similar if $y \in f(A_2)$. In either case, $y \in \text{LHS}$.

(b) (\subseteq) Pick $y \in f(A_1 \cap A_2)$. This means $y = f(x)$ for some $x \in A_1 \cap A_2$. Knowing $x \in A_1$ gives $y \in f(A_1)$; knowing $x \in A_2$ gives $y \in f(A_2)$. Since we know both, $y \in f(A_1) \cap f(A_2)$, as req'd.

(c) (\supseteq) Pick $y \in f(A_1) \cap f(A_2)$. This means $y = f(x_1)$ for some $x_1 \in A_1$ and $y = f(x_2)$ for some $x_2 \in A_2$. Now $f(x_1) = f(x_2)$ and since f is 1-1, this implies $x_1 = x_2$. This point is in both A_1, A_2 i.e. $x_1 \in A_1 \cap A_2$. Hence $y \in f(A_1 \cap A_2)$.

- [8] 4. Prove that $\lim_{n \rightarrow \infty} (\sqrt{n^2+1} - n) = 0$. Use the ε -based definition, not limit laws!

$$\begin{aligned} \text{Note } \sqrt{n^2+1} - n &= \frac{\sqrt{n^2+1} - n}{1} \cdot \left(\frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n} \right) = \frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n} \\ &= \frac{1}{\sqrt{n^2+1} + n}. \end{aligned}$$

$$\text{Therefore } \left| \sqrt{n^2+1} - n \right| = \frac{1}{\sqrt{n^2+1} + n} \leq \frac{1}{2n}.$$

$$\text{Observe } \frac{1}{2n} < \varepsilon \iff \frac{1}{2\varepsilon} < n.$$

PROOF: Given $\varepsilon > 0$, define $N = \left\lceil \frac{1}{2\varepsilon} \right\rceil$.

Then for each $n > N$,

$$\left| a_n \right| < \frac{1}{2n} < \varepsilon.$$

Nice alt trick:

$$n^2 + 1 < n^2 + 2 + \frac{1}{n^2} = \left(n + \frac{1}{n}\right)^2$$

$$\Rightarrow \left| \sqrt{n^2+1} - n \right| = \sqrt{n^2+1} - n < \sqrt{\left(n + \frac{1}{n}\right)^2} - n = \frac{1}{n}.$$

[8] 5. Let $A = \{a_1, a_2, \dots\}$. Define $B = A - \{a_{n^2} : n \in \mathbb{N}\}$.

- (a) Assume, for this part only, that $j \neq k$ implies $a_j \neq a_k$. Prove that $|A| = |B|$.
 (b) Prove or disprove: If $|A| = |\mathbb{N}|$, then $|A| = |B|$.

(a) Clearly $|A| = \aleph_1$.

Also, B is an infinite set with $B \subseteq A$.
 (Why infinite? b/c $B \supseteq \{a_2, a_5, a_{10}, \dots, a_{n^2+1}, \dots\}$ and RHS is infinite.)

Hence B is infinite and countable, giving $|B| = |\mathbb{N}| = |A|$.

(b) Stmt is false. When list a_1, a_2, \dots allows redundancy, we can have a situation like this:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2, a_3 = 2, a_4 = 2 \\ a_5 &= 3, a_6 = 3, a_7 = 3, a_8 = 3, a_9 = 3, \\ a_{10} &= 4, \dots, a_{16} = 4, \\ &\text{etc.} \end{aligned}$$

Here the recipe says $a_j = m$ whenever $(m-1)^2 < j \leq m^2$.
 It gives $\{a_1, a_4, a_9, a_{16}, \dots\} = \{1, 2, 3, 4, \dots\} = A$, so

$$B = A - \{a_1, a_4, a_9, a_{16}, \dots\} = \emptyset.$$

Clearly $|B| = 0$ differs from $|A| = \aleph_0$.

[10] 6. In each sentence below, write a precise definition for the term or statement in the box.

(a) The function $f: A \rightarrow B$ is onto.

$$\forall y \in B \exists x \in A : y = f(x).$$

(b) Statements P and Q are given, and $P \Rightarrow Q$.

$$Q \vee (\sim P).$$

(c) A sequence (a_n) is given, and $a_n \rightarrow 8$ as $n \rightarrow \infty$.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n > N, |a_n - 8| < \varepsilon.$$

(d) A bounded nonempty set $S \subseteq \mathbb{R}$ is given, and $\alpha = \inf(S)$.

(i) $\forall x \in S, \alpha \leq x$

(ii) $\forall \varepsilon > 0 \exists x \in S : x < \alpha + \varepsilon$

(e) Nonempty sets A , B , and C are given, with $C \subseteq A$. Define $g(C)$, where g is a relation from A to B .

$$g(C) = \left\{ y \in B : \exists x \in C \text{ where } (x, y) \in g \right\}$$

- (f) The set of real numbers, \mathbb{R} , has the least upper bound property.

If $S \subseteq \mathbb{R}$ is bounded above and nonempty
there exists a real number $\beta = \sup(S)$.

- (g) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$.

Sequence $(s_N)_{N \in \mathbb{N}}$ defined by $s_N = \sum_{n=1}^N \frac{1}{n^2}$ obeys

$$\forall \epsilon > 0 \exists N_0 \in \mathbb{N} \text{ s.t. } \forall N > N_0, |s_N - \frac{\pi^2}{6}| < \epsilon.$$

- (h) The function $f: A \rightarrow B$ is not a bijection.

Either f is not 1-1 or f is NOT ONTO.
(Inclusive or is assumed throughout mathematics.)

- (i) Statements P and Q are given, and $P \Rightarrow Q$ is true but vacuous.

P is false

- (j) Max's favourite integers are the ones satisfying $x \equiv 2 \pmod{7}$.

$$\frac{x-2}{7} \in \mathbb{Z}$$

OR, $\exists n \in \mathbb{Z}$ s.t. $x = 7n + 2$.

[9] 7. For each statement given below, do two things:

- (i) Write the negation as a simple direct statement.
(For example, just writing "The following is false" in front of the statement is not acceptable.)
- (ii) Prove either the given statement or its negation, as appropriate.

(a) For each $n \in \mathbb{N}$, $3^n + 2$ is prime.

(i) $(\sim a)$: There exists $n \in \mathbb{N}$ such that $3^n + 2$ is composite.

(ii) $n=5$ confirms $(\sim a)$, b/c $3^5 + 2 = 243 + 2 = 245$ is divisible by 5.

(b) There exists an integer n such that $n^2 + 3n/2 = 1$.

(i) $(\sim b)$: For each integer n , $n^2 + \frac{3}{2}n \neq 1$

(ii) Stmt (b) is true; $n = -2$ gives

$$n^2 + \frac{3}{2}n = 4 - 3 = 1.$$

(c) If $x/(x-1) \leq 2$, then $x < 1$ or $x \geq 2$.

(i) $(\sim c)$: There exists x such that $\frac{x}{x-1} \leq 2$

and $x \geq 1$ and $x < 2$.

(ii) (c) is true; b/c $(\sim c)$ is impossible. If x obeys $x \geq 1$ and $\frac{x}{x-1} \leq 2$, then $x \leq 2(x-1)$ forces $x \geq 2$.

CONTRADICTION: Assume $x \in [1, 2)$. If $x=1$, $\frac{x}{x-1}$ is undefined, so " $\frac{x}{x-1} \leq 2$ " is FALSE. So far, so good. If $1 < x < 2$, then $1 > x-1 > 0$

so $\frac{1}{x-1} > 1$ so $\frac{1}{x-1} + \frac{x-1}{x-1} > 2$, i.e., $\frac{x}{x-1} > 2$. Done.

- [9] 8. (a) Prove: For real numbers a and c , one has

$$[\forall \varepsilon > 0, a \leq c + \varepsilon] \implies [a \leq c].$$

- (b) Prove: Whenever A and B are bounded nonempty subsets of \mathbb{R} ,
 $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

(a) Contrapositive says: $a > c \implies \left(\exists \varepsilon > 0 \text{ s.t. } a > c + \varepsilon \right)$.

This is nearly obvious: given any $a, c \in \mathbb{R}$ with $a > c$,
 define $\varepsilon = \frac{a-c}{2}$. Clearly $\varepsilon > 0$, and $c + \varepsilon = \frac{a+c}{2} < a$.

(b) Let $\alpha = \sup(A)$, $\beta = \sup(B)$, $\mu = \max\{\alpha, \beta\}$.

- We know: $\left. \begin{array}{l} \forall x \in A, x \leq \alpha \\ \forall x \in B, x \leq \beta \end{array} \right\} \therefore \forall x \in A \cup B, x \leq \max\{\alpha, \beta\} = \mu$.

So μ is an upper bound for $A \cup B$.

- Given any $\varepsilon > 0$, let's show $\mu - \varepsilon$ is NOT an upper bound.

Case 1: $\mu = \alpha$.

We know $\exists x \in A$ s.t. $\alpha - \varepsilon < x$. This same x
 obeys $x \in A \cup B$ and $\mu - \varepsilon < x$, as req'd.

Case 2: $\mu = \beta$.

Here $\exists x \in B$ s.t. $\beta - \varepsilon < x$. This same x
 obeys $x \in A \cup B$ and $\mu - \varepsilon < x$, as req'd.

The two bulleted stories confirm the defⁿ of $\mu = \sup(A \cup B)$.

[9] 9. Prove: $\forall n \in \mathbb{N}, \underbrace{1 \cdot 3^0 + 2 \cdot 3^1 + 3 \cdot 3^2 + \dots + n \cdot 3^{n-1}} = \frac{1 + (2n-1)3^n}{4}$.

call this $P(n)$

$$P(1): 1 \cdot 3^0 = \frac{1 + (1)3^1}{4} \text{, i.e., } 1=1. \text{ TRUE.}$$

Assume, for some $n \in \mathbb{N}$, $P(n)$.

$$P(n): 1 \cdot 3^0 + \dots + n \cdot 3^{n-1} = \frac{1 + (2n-1)3^n}{4}$$

Try to prove $P(n+1)$, namely,

$$1 \cdot 3^0 + \dots + n \cdot 3^{n-1} + (n+1) \cdot 3^n = \frac{1 + (2n+1)3^{n+1}}{4}$$

So add $(n+1)3^n$ to both sides of known fact $P(n)$:

$$1 \cdot 3^0 + \dots + n \cdot 3^{n-1} + (n+1)3^n = \frac{1 + (2n-1)3^n}{4} + (n+1)3^n$$

$$= \frac{1 + (2n-1)3^n}{4} + \frac{4(n+1)3^n}{4}$$

$$= \frac{1 + 3^n [2n-1 + 4n+4]}{4}$$

$$= \frac{1 + 3^n [6n+3]}{4}$$

$$= \frac{1 + [2n+1]3^{n+1}}{4}$$

That worked!

[10] 11. Recall that a sequence $(a_n)_{n \in \mathbb{N}}$ is called "Cauchy" when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall m, n > N, |a_m - a_n| < \varepsilon.$$

Prove: If (a_n) is a Cauchy sequence and one has $\lim_{n \rightarrow \infty} b_k = 220$, where the sequence (b_k) is defined by

$$b_1 = a_2, b_2 = a_4, b_3 = a_8, \dots, b_k = a_{2^k}, \dots,$$

then $a_n \rightarrow 220$ as $n \rightarrow \infty$.

Our goal: Prove that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |a_n - 220| < \varepsilon.$$

Key idea: $|a_n - 220| \leq |a_n - a_{2^k}| + |a_{2^k} - 220| \quad \forall k, n$

and both terms on RHS can be controlled.

Proof: Given $\varepsilon > 0$ arb, define $\varepsilon' = \frac{\varepsilon}{3}$ and get $N' \in \mathbb{N}$ from Cauchy satisfying

$$\forall m, n > N', |a_m - a_n| < \varepsilon' = \frac{\varepsilon}{3}.$$

This N' achieves our goal. Indeed, each $n > N'$ obeys $\otimes |a_n - 220| \leq |a_n - a_{2^k}| + |a_{2^k} - 220| \quad \forall k \in \mathbb{N}$.

Since $b_k = a_{2^k} \rightarrow 220$, there exists $K' \in \mathbb{N}$

so large that $\forall k > K', |b_k - 220| < \varepsilon'$. Focus on some $k > K'$ such that $m = 2^k > N'$. Then, from \otimes ,

$$|a_n - 220| \leq \varepsilon' + \varepsilon' = \frac{2}{3}\varepsilon < \varepsilon$$

(recall $n > N'$), as required.

[9] 12. Determine whether each series converges or diverges. Justify your answer.

$$(a) \sum_{n=1}^{\infty} \frac{\sqrt{n+9} - \sqrt{n}}{n}$$

$$0 \leq a_n = \frac{\sqrt{n+9} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+9} + \sqrt{n}}{\sqrt{n+9} + \sqrt{n}} = \frac{9}{n(\sqrt{n+9} + \sqrt{n})} \leq \frac{9}{2n^{3/2}}$$

So this series CONVERGES [ABSOLUTELY] by comparison with the p -series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

$$(b) \sum_{n=1}^{\infty} \frac{n^5}{5^n} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^5 / 5^{n+1}}{n^5 / 5^n} \right| = \left(\frac{n+1}{n} \right)^5 \cdot \frac{5^n}{5^{n+1}} = \left(1 + \frac{1}{n} \right)^5 \cdot \frac{1}{5}$$

As $n \rightarrow \infty$, this ratio approaches $\beta = 1/5$, so

the series CONVERGES [ABSOLUTELY] by the ratio test.
(The root test works well, too.)

$$(c) \sum_{n=1}^{\infty} \frac{n^2}{2009n^2 + 10^{10}}$$

$$\text{Here } a_n = \frac{n^2}{2009n^2 + 10^{10}} = \frac{1}{2009 + \frac{10^{10}}{n^2}} \rightarrow \frac{1}{2009}$$

as $n \rightarrow \infty$. This series DIVERGES, by the Crude Test.