MAT 2122

Solutions to assignment #3

1. Here, $\vec{a} = (0,0)$, so $f(\vec{a}) = f(0,0) = \frac{1}{0+0+1} = 1$.

$$\nabla f = \left(\frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2} \right),$$

so $\nabla f(\vec{a}) = (0,0)$

As for second partial derivatives,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-2x}{(x^2+y^2+1)^2}\right) = \frac{(-2)(x^2+y^2+1)^2 - (-2x)(x^2+y^2+1)}{(x^2+y^2+1)^4}$$

$$= \frac{-2(x^2+y^2+1)^2 - (-2x)(2)(x^2+y^2+1)(2x)}{(x^2+y^2+1)^4}$$

$$= \frac{-2(x^2+y^2+1) + 8x^2}{(x^2+y^2+1)^3} = \frac{6x^2-2y^2-2}{(x^2+y^2+1)^3},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-2y}{(x^2+y^2+1)^2}\right) = \frac{\partial}{\partial x}\left(-2y(x^2+y^2+1)^{-2}\right)$$

$$= -2y(-2)(x^2+y^2+1)^{-3}(2x)$$

$$= 8xy(x^2+y^2+1)^{-3},$$

and finally

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{6y^2-2x^2-2}{(x^2+y^2+1)^3}$$

(obtained either directly, or using symmetry between x and y in the formula for f)

So $\frac{\partial^2 f}{\partial x^2}(\vec{a}) = \frac{-2}{1} = -2$

$$\frac{\partial^2 f}{\partial x \partial y}(\vec{a}) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$ by Theorem 3.1

$$\frac{\partial^2 f}{\partial y^2}(\vec{a}) = \frac{-2}{1} = -2.$$

So the quadratic approximation is (writing $\vec{a} = (a, b)$)

$$Q(x, y) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (x-a, y-b)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(\vec{a})(x-a)^2 + \frac{\partial^2 f}{\partial x \partial y}(\vec{a})(x-a)(y-b)\right.$$
$$\left. + \frac{\partial^2 f}{\partial y \partial x}(\vec{a})(y-b)(x-a) + \frac{\partial^2 f}{\partial y^2}(\vec{a})(y-b)^2\right)$$

$$= 1 + (0,0) \cdot (x-0, y-0)$$
$$+ \frac{1}{2}\left(-2(x-0)^2 + 0 + 0 + (-2)(y-0)^2\right)$$

$$= 1 - x^2 - y^2.$$

The other form we've used for the quadratic approximation is

$$Q(\vec{a} + \vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \frac{1}{2}\left(\sum_{i,j=1}^{2} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})\right),$$

which becomes

$$Q((0,0) + (h_1, h_2)) = 1 - h_1^2 - h_2^2,$$

$$Q(h_1, h_2) = 1 - h_1^2 - h_2^2,$$ the same as above.

#2 (a) Here $\nabla f = (2x+3y,\ 2y+3x)$ is defined everywhere

To solve $\nabla f = (0,0)$, we write out the equations

$$(1) \quad 2x+3y=0$$
$$(2) \quad 3x+2y=0$$
(we re-ordered to have x's first)

$3\cdot(1) - 2\cdot(2)$ gives $5y=0$, so $y=0$.
Plugging back into (1) gives $x=0$.

So the only critical point is $(0,0)$.

To determine local max/min or saddle, we need the second derivatives at the critical point. Let's first write the formulas for the second derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x+3y) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x+2y) = 3, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x+2y) = 2.$$

They are constant, so when we evaluate at $\vec{a} = (0,0)$, we get these same values.

Now, $$D = \left(\frac{\partial^2 f}{\partial x^2}(\vec{a})\right)\left(\frac{\partial^2 f}{\partial y^2}(\vec{a})\right) - \left(\frac{\partial^2 f}{\partial x \partial y}(\vec{a})\right)^2$$

$$= 2 \cdot 2 - 3^2$$

$$= -5 < 0.$$

Since $D<0$, we have a saddle point.

So there is only one critical point, $(0,0)$, and it is a saddle point.

(b) Here, $\nabla f = \left(1\cdot(xy+1) + (x+y)\cdot y,\ 1\cdot(xy+1) + (x+y)\cdot x\right)$,

again defined everywhere. Let's solve $\nabla f = (0,0)$.

$$xy+1 + (x+y)\cdot y = 0 \quad \rightsquigarrow \quad 2xy + y^2 + 1 = 0 \quad (1)$$

$$xy+1 + (x+y)\cdot x = 0 \quad \rightsquigarrow \quad 2xy + x^2 + 1 = 0. \quad (2)$$

Taking (1) − (2) gives $y^2 - x^2 = 0$, $(y-x)(y+x) = 0$

So $\cancel{y} = x$ or $y = -x$. We consider these cases separately.

If $y = x$, then (1) becomes $3x^2 + 1 = 0$, $x = \pm\sqrt{\frac{-1}{3}}$, not real.
no solution.

If $y = -x$, then (1) becomes
$$2x(-x) + (-x)^2 + 1 = 0$$
$$-x^2 + 1 = 0$$
$$x = \pm 1.$$

If $x = 1$, $y = -x = -1$
If $x = -1$, $y = 1$

So the only critical points are $(1,-1)$ and $(-1,1)$.

Now let's find the second derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2xy + y^2 + 1\right) = 2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2xy + x^2 + 1\right) = 2y + 2x = 2x + 2y, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(2xy + x^2 + 1\right) = 2x$$

At $(1,-1)$, $\frac{\partial^2 f}{\partial x^2}(1,-1) = -2$, $\frac{\partial^2 f}{\partial x \partial y}(1,-1) = 0$,

$\frac{\partial^2 f}{\partial y^2}(1,-1) = 2$, and

$$D = \left(\frac{\partial^2 f}{\partial x^2}(1,-1)\right)\left(\frac{\partial^2 f}{\partial y^2}(1,-1)\right) - \left(\frac{\partial^2 f}{\partial x \partial y}(1,-1)\right)^2 = (-2)(2) - 0^2 = -4 < 0.$$

So $(1,-1)$ is a saddle point.

At $(-1,1)$, $D = (2)(-2) - 0^2 = -4 < 0$, so $(-1,1)$ is also a saddle point too.

So there are two critical points $(1,-1)$, $(-1,1)$ and both are saddle points.

(Notice that $f(1.1,-1)$ is smaller than $f(1,-1)$, and

$f(1,-1.1)$ is larger than $f(1,-1)$,

which is further confirmation that $(1,-1)$ is <u>not</u> a local max or a local min of f.)

3 (a) Here we take $g_1(x,y,z) = x^2 + 2y^2 = 2 = c_1$

$$g_2(x,y,z) = x + y - z = 0 = c_2$$

At an extreme point, we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$g_1 = c_1$$

$$g_2 = c_2$$

This becomes $\frac{\partial f}{\partial x} = \lambda_1 \frac{\partial g_1}{\partial x} + \lambda_2 \frac{\partial g_2}{\partial x}$, and similarly with $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$.

So we have

$$4x - 1 = \lambda_1(2x) + \lambda_2(1)$$

$$1 = \lambda_1(4y) + \lambda_2(1)$$

$$1 = \lambda_1(0) + \lambda_2(-1)$$

The last equation gives $\lambda_2 = -1$; plugging into the first gives

$$4x - 1 = 2\lambda_1 x - 1 \quad , \quad 2(\lambda_1 - 2)x = 0.$$

So $\lambda_1 = 2$ or $x = 0$.

In the case $x=0$, the first constraint implies $0^2 + 2y^2 = 2$, $y = \pm 1$.
The second constraint then gives $0 + y - z = 0$, $z = y$

~~Either~~

So we have two points: $(0,1,1)$ and $(0,-1,-1)$, if $x=0$

In the case $\lambda_1 = 2$, we get, from the second equation,

$$1 = 2(4y) - 1, \qquad y = \frac{1}{4}.$$

Plugging into the first constraint gives $x^2 + 2\left(\frac{1}{4}\right)^2 = 2$,

$$x = \pm\sqrt{2 - \frac{1}{8}} = \pm\sqrt{\frac{15}{8}}.$$

Plugging into the second constraint gives $z = x + y = \pm\sqrt{\frac{15}{8}} + \frac{1}{4}$.

We end up with two more points, $\left(\sqrt{\frac{15}{8}}, \frac{1}{4}, \sqrt{\frac{15}{8}} + \frac{1}{4}\right)$

$\left(-\sqrt{\frac{15}{8}}, \frac{1}{4}, -\sqrt{\frac{15}{8}} + \frac{1}{4}\right)$.

We compare the value of f at these four points:

(x, y, z)	$f(x, y, z)$
$(0, 1, 1)$	2
$(0, -1, -1)$	-2
$\left(\sqrt{\frac{15}{8}}, \frac{1}{4}, \sqrt{\frac{15}{8}} + \frac{1}{4}\right)$	$2 \cdot \frac{15}{8} \cancel{- \sqrt{\frac{15}{8}}} + \frac{1}{4} \cancel{+ \sqrt{\frac{15}{8}}} + \frac{1}{4} = \frac{17}{4}$
$\left(-\sqrt{\frac{15}{8}}, \frac{1}{4}, -\sqrt{\frac{15}{8}} + \frac{1}{4}\right)$	$2 \cdot \frac{15}{8} \cancel{+ \sqrt{\frac{15}{8}}} + \frac{1}{4} \cancel{- \sqrt{\frac{15}{8}}} + \frac{1}{4} = \frac{17}{4}$

So there is a max. at $\left(\pm\sqrt{\frac{15}{8}}, \frac{1}{4}, \pm\sqrt{\frac{15}{8}} + \frac{1}{4}\right)$, and a min at $(0, -1, -1)$.

Since the region is an ellipse (the intersection of a plane with a vertical elliptical cylinder) it is bounded, so f

does achieve its max and min, so the above really are the global max and min of the function

(b) Now we have $g(x,y) = 2x^2+2y$, with the constraint $h(x,y) = x^2+2y^2 = 2$.

Using Lagrange multipliers,

$$\begin{aligned} 4x &= 2\lambda x \\ 2 &= 4\lambda y \end{aligned}$$

$$x^2+2y^2=2$$

As before, we can reduce this to four points: (x,y) can equal $(0,1)$, $(0,-1)$, $\left(\sqrt{\frac{15}{8}}, \frac{1}{4}\right)$, $\left(-\sqrt{\frac{15}{8}}, \frac{1}{4}\right)$, and again the last two points are global max, and the point $(0,-1)$ is a global min.

4 We apply the inverse function theorem.

$$F(x,y,z) = (u,v,w) = (x+xyz,\ y+xy,\ z+2x+3z^2)$$

The Jacobian of F at $(0,0,0)$ is

$$\det \begin{bmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{bmatrix}(0,0,0) = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = 1.$$

Since the Jacobian at $(x,y,z)=(0,0,0)$ is non zero,

the inverse function theorem implies that the system can be

solved for (x,y,z), when (u,v,w) is close to $F(0,0,0)=(0,0,0)$.