2141.F08 Ass 3 Soln



**1.** Let $\mathcal{P}(\mathbf{R})$ be the real vector space of polynomial functions, and define a map $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by

$$T(p) = p' - p.$$

(Here $p'$ denotes the derivative of $p$.)

a) Prove that $T$ is linear. (2)

b) Prove that $T$ is injective. (3)

a) Let $c_1, c_2 \in \mathbb{R}$ and $p_1, p_2 \in \mathcal{P}(\mathbb{R})$. Then

$$\begin{aligned} T(c_1 p_1 + c_2 p_2) &= (c_1 p_1 + c_2 p_2)' - (c_1 p_1 + c_2 p_2) \\ &= c_1 p_1' + c_2 p_2' - c_1 p_1 - c_2 p_2 \quad \text{(differentiation is linear)} \\ &= c_1 (p_1' - p_1) + c_2 (p_2' - p_2) \\ &= c_1 T(p_1) + c_2 T(p_2) \end{aligned}$$

Hence, $T$ is linear.

b) Let $p(t) = a_0 + \cdots + a_n t^n \in \mathcal{P}(\mathbb{R})$. Then $T(p) = 0 \iff$
$\forall t \in \mathbb{R}, T(a_0 + a_1 t + \cdots + a_n t^n) = a_1 - a_0 + (2a_2 - a_1)t + (3a_3 - a_2)t^2 + \cdots + (-a_n)t^n = 0$

Since $\{1, t, \ldots, t^n\}$ is l.i. (for any $n \in \mathbb{N}$), $a_1 - a_0 = 0 = 2a_2 - a_1 = \cdots = -a_n$ $(*)$

This is the homogeneous linear system in the $n+1$ variables $a_0, \ldots, a_n$ whose coefficient matrix is

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 3 & 0 \cdots & 0 \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & -1 & n \\ 0 & & \cdots & & 0 & -1 \\ 0 & & & & & \end{bmatrix} = [c_{ij}], \quad \text{where } c_{ij} = \begin{cases} -1, & i=j \\ 1, & j=i+1 \\ 0, & \text{else.} \end{cases}$$

This matrix clearly has rank $n+1$, and so the only soln to $(*)$ is $a_0 = a_1 = \cdots = a_n = 0$. Hence $p = 0$. Thus $\ker T = \{0\}$, and so $T$ is injective.

(1) Some idea that would work
+ (1) some development towards success
0, ½, 1 + (1) a well written, correct solution.

**2.** Recall that for any vector space $U$ over a field $\mathbf{F}$, $\mathcal{L}(U, \mathbf{F})$ is called the dual space of $U$ and is denoted $U^*$.

Suppose $T : V \to W$ is a linear map.

2 a) Let $f \in W^*$. Show that the formula $g(v) = f(T(v))$ defines an element $g \in V^*$

Now define a map $\tilde{T} : W^* \to V^*$ by $\tilde{T}(f) = g$, where $g$ is defined in (a). That is, for $f \in W^*$, $\tilde{T}(f)$ is the member of $V^*$ which defined by

$$\tilde{T}(f)(v) = f(T(v)), \forall v \in V.$$

3 b) Prove that $\tilde{T}$ is a linear map.

a) The formula clearly shows that $g : V \to F$. To see that $g$ is linear, note that the formula shows that $g = f \circ T$ is the composition of 2 linear maps ($T$ & $f$), and so is linear. Hence $g \in V^*$.

b) Let $f_1, f_2 \in W^*$ and $c_1, c_2 \in F$. Then, $\forall v \in V$

$$\begin{aligned} \tilde{T}(c_1 f_1 + c_2 f_2)(v) &= (c_1 f_1 + c_2 f_2)(T(v)) \\ &= c_1 f_1 T(v) + c_2 f_2 T(v) && \text{(by defn of add}^n \text{ \& mult}^n \text{ by scalars in } W^*) \\ &= c_1 \tilde{T}(f_1)(v) + c_2 \tilde{T}(f_2)(v) && (\text{def}^n \text{ of } \tilde{T}) \\ &= (c_1 \tilde{T}(f_1) + c_2 \tilde{T}(f_2))(v) && (\text{def}^n \text{ of add}^n \text{, mult}^n \text{ by scalars in } V^*) \end{aligned}$$

$$\therefore \quad \tilde{T}(c_1 f_1 + c_2 f_2) = c_1 \tilde{T}(f_1) + c_2 \tilde{T}(f_2). \quad \text{(as functions)}$$

Hence $\tilde{T}$ is linear.

2 knowing what "$\tilde{T}$ is linear" means  1 a correct sol$^n$, reasonably presented

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2c) If $V = \mathbf{R}^2, W = \mathbf{R}^3$, $T : \mathbf{R}^2 \to \mathbf{R}^3$ is multiplication by the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, and $g \in (\mathbf{R}^3)^*$ is defined by $g(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = x + y - z$, find an explicit formula for $\tilde{T}(g) \in (\mathbf{R}^2)^*$, and use it to compute $\tilde{T}(g)(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$

By defn,

$$\tilde{T}(g)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = g\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = g\left(\begin{bmatrix} x+2y \\ 3x+4y \\ 5x+6y \end{bmatrix}\right)$$

(by defn of $T$)

(by defn of $\tilde{T}$)

(by the defn of $g$) $= (x+2y) + (3x+4y) - (5x+6y)$

$= -x.$

i.e. $\tilde{T}(g)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -x.$

Hence, $\tilde{T}(g)\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = -1.$

1 formula for $\tilde{T}(g)\begin{bmatrix} x \\ y \end{bmatrix}$

1 correct answer for $\tilde{T}(g)(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$

1 well written justification

If $\tilde{T}(g)\begin{bmatrix} x \\ y \end{bmatrix}$ and $\tilde{T}(g)(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$ are correct but there is no justn, 0/3

**3.** Suppose $T : U \to V$ and $S : V \to W$ are linear maps.

3 a) Prove that if $T$ is surjective, then $\mathrm{im}\,(ST) = \mathrm{im}\ S$ — 1 $\subseteq$ 1 $\supseteq$ 1 for knowing to do it

3 b) Prove that if $S$ is injective, then $\ker T = \ker(ST)$ — //

3 c) Find two linear maps $S, T : \mathbf{R}^2 \to \mathbf{R}^2$ such that
(i) $\mathrm{im}\,(ST) \neq \mathrm{im}\ S$, and
(ii) $\ker T \neq \ker(ST)$

1 - example for (i)
1 - possibly different example for (ii)
+1 - same example for (i) & (ii)

a) If $w \in \mathrm{im}\, ST$, then $\exists\, u \in U$ with $w = ST(u) = S(T(u))$, so $w \in \mathrm{im}\, S$. So $\mathrm{im}(ST) \subseteq \mathrm{im}\, S$. Given $w' \in \mathrm{im}\, S$, $\exists\, v' \in V$ with $w' = S(v')$. As T is surjective, $\exists\, u' \in U$ with $T(u') = v'$. Hence $w' = S(v') = S(T(u')) = ST(u')$, and so $w' \in \mathrm{im}\, ST$. Hence $\mathrm{im}\, S \subseteq \mathrm{im}\, ST$ and so $\mathrm{im}\, S = \mathrm{im}\, ST$. (Note that $\mathrm{im}(ST) \subseteq \mathrm{im}\, S$ is always true, for any linear $T : U \to V$, $S : V \to W$.)

b) If $u \in \ker T$, then $Tu = 0$, and so $ST(u) = S(T(u)) = S(0) = 0$. Hence $u \in \ker T \Rightarrow u \in \ker ST$. That is, $\ker T \subseteq \ker ST$. $(*)$
Given $u' \in \ker ST$, we know $0 = ST(u') = S(T(u'))$. Hence $T(u')$ belongs to the kernel of $S$, i.e. $T(u') \in \ker S$. But $\ker S = \{0\}$, as $S$ is injective, so $T(u') = 0$. This of course means that $u' \in \ker T$. Hence, we've shown that $\ker ST \subseteq \ker T$. Together with $(*)$, this means that $\ker ST = \ker T$.

(Note that $\ker T \subseteq \ker ST$, for any $S : V \to W$ and $T : U \to V$.)

c) Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be the matrices of the linear maps they represent. Then $ST = 0$, but neither $S = 0$ nor $T = 0$. Hence

(i) $\mathrm{im}(ST) = \{0\} \neq \mathrm{im}\, S = \mathrm{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

(ii) $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \ker T \neq \mathbb{R}^2 = \ker ST$.

**4.** Let $T : V \to V$ be a linear map such that $T^2 = T$, and let $I$ denore the identity map of $V$. (i.e. $I(v) = v, \forall v \in V$.) Prove the following:

a) $\operatorname{im} T = \ker(T - I)$ 2 1 $\subseteq$ 1 $\supseteq$.

b) $V = \operatorname{im} T \oplus \ker T$ 3 1 knowing what to do + 1 some progress
0, ½, 1 well written & correct

a) Let $v \in \operatorname{im} T$. Then $v = T(v')$ for some $v' \in V$, and so $(T-I)v = (T-I)T(v') = T^2(v') - T(v') = (T^2 - T)(v') = 0$. Hence $v \in \ker(T-I)$. If $v \in \ker(T-I)$, then $Tv = v$, and so $v \in \operatorname{im} T$. Hence $\operatorname{im} T \subseteq \ker(T-I)$, and $\ker(T-I) \subseteq \operatorname{im} T$,

and so $\operatorname{im} T = \ker(T-I)$.

b) Note that for any $v \in V$, $v = Tv + v - Tv$. Clearly, $T(v) \in \operatorname{im} T$, and $T(v - Tv) = Tv - T^2 v = 0$; since $T = T^2$. Hence any $v$ can be written as a sum of vectors, one in $\operatorname{im} T$, and the other in $\ker T$. i.e. $V = \operatorname{im} T + \ker T$. $-(1)$

Now suppose $v \in \operatorname{im} T \cap \ker T$. Then $\exists v' \in V$ s.t. $T(v') = v$; i.e. $v \in \operatorname{im} T \cap \ker T$. As $v \in \ker T$, $0 = T(v) = T^2(v') = T(v') = v$. $\Rightarrow v = 0$. Hence $\operatorname{im} T \cap \ker T = \{0\}$. (2). Properties (1) & (2) together show that $V = \operatorname{im} T \oplus \ker T$.

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**5.** (Bonus) Prove that the linear map $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ defined in question 1 is in fact an isomorphism.

Proof (1). If $C_n = [c_{ij}]$ is the matrix of Q1 (b), we know that if $v = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$, $w = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = C_n v$, then

$$T(a_0 + a_1 t + \cdots + a_n t^n) = b_0 + b_1 t + \cdots + b_n t^n.$$

Hence, if $\forall n \in \mathbb{N}, \forall w = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^{n+1}$, $\exists v = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n+1}$ such that $C_n v = w$, this shows that for every $n \in \mathbb{N}$, and every polynomial $q(t) = b_0 + \cdots + b_n t^n$ of degree $n$, there is a polynomial $p(t) = a_0 + \cdots + a_n t^n$ such that $T(p) = q$.

That is, if $\forall n \in \mathbb{N}$, $C_n v = w$ is consistent for every $w \in \mathbb{R}^{n+1}$, then $\forall n \in \mathbb{N}$, and $\forall q \in \mathcal{P}(\mathbb{R})$ of degree $n$, $\exists p \in \mathcal{P}(\mathbb{R})$ (coincidentally of degree $n$) s.t. $T(p) = q$. Hence $T$ is surjective, iff $\forall n \in \mathbb{N}$, the linear system $C_n v = w$ is consistent for every $w \in \mathbb{R}^{n+1}$. But we know from 1 (b), that, $\forall n \in \mathbb{N}$, rank $C_n = n+1$, which implies that $C_n v = w$ is consistent for every $w \in \mathbb{R}^{n+1}$. Hence, $T$ is surjective.

Thus, with 1 (a) & (b) this shows that $T$ is an isomorphism.

See the next page for another pf

Pf (2) Since $T = D - I$, where $D(p) = p'$,

$T^{-1}$, if it exists would be $-(I-D)^{-1}$. But we know from high school that $\frac{1}{1-x} = \sum_{k\geq 0} x^k = 1 + x + x^2 + \cdots$ (at least if $|x|<1$) while this apparently makes no sense here, we press on. Note that for any $p \in \mathcal{P}(\mathbb{R})$, if $\deg p = n$, then $D^{n+1} = 0$, and so if we were to write

$S_n = I + D + D^2 + \cdots + D^n$, and apply $S$ to a poly. of degree $n$, we'd find that $(I-D)S_n(p) =$

$(I-D)(I + D + D^2 + \cdots + D^n)(p) = I(p) - D^{n+1}(p) = p$.

i.e. for $p$ of degree $n$, $(I-D)S_n(p) = p$. A similar calcul<sup></sup>n shows $S_n(I-D)(p) = p$, if $\deg(p) \leq n$.

This shows that if $S = \sum_{k=0}^{\infty} D^k$, then, despite appearances, $S : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is well-defined (all polys $p$ in $\mathcal{P}(\mathbb{R})$ have some finite degree $m$, and then $S(p) = S_m(p)$) and is linear. Moreover, if $\deg p = n$,

$(I-D)S(p) = (I-D)S_m(p) = p$, which shows that

$-TS(p) = p$, or $TS(-p) = p$. This shows that $T$ is surjective, and with 1(a), (b), demonstrates that $T$ is an isomorphism.