University of Ottawa

Department of Mathematics and Statistics

MAT 3320 : Mathematics for Engineers
Professor: Eric Hua

Midterm Exam

Thursday, June 20th

Surname ____________________ First Name ____________________

Student # ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper.
(e) Students will be allowed to use basic scientific calculators like the TI-30XXs on the midterm and final exam. Graphing and/or programmable calculators and calculators with integration/ differentiation capabilities will not be allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	Total
Maximum	2	2	2	2	2	4	6	5	25
Grade									

1. Which of the following is a solution of $x^2y''(x) + xy'(x) + (4x^2 - 3)y(x) = 0$, for $x > 0$:

(a) $J_2(\sqrt{3}x)$ (b) $J_4(\sqrt{3}x)$ (c) $Y_3(4x)$ (d) $J_{\sqrt{3}}(2x)$ (e) $J_{\sqrt{3}}(4x)$

Solution: (d). Note that $\lambda^2 = 4$ and $\nu^2 = 3$, $\Rightarrow \lambda = 2$ and $\nu = \sqrt{3}$. Hence

$$y_1(x) = J_{\sqrt{3}}(2x), \quad y_2(x) = J_{-\sqrt{3}}(2x).$$

2. Which of the following is a solution of $(1 - x^2)y'' - 2xy' + 30y = 0$:

(a) $P_1(x)$ (b) $P_2(x)$ (c) $P_3(x)$ (d) $P_4(x)$ (e) $P_5(x)$

Solution: (e).

$$n(n+1) = 30 \Rightarrow n = 5, -6, \Rightarrow y_1 = P_5(x).$$

3. Let $f(x) = 7x^4$, $-1 < x < 1$. The Fourier-Legendre expansion is

$$f(x) \approx c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + c_4P_4(x) + \cdots .$$

Find c_0.

(a) 3.3 (b) 1.4 (c) 2.6 (d) 3.4 (e) 5.6

Solution: (b).
From $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ we imply that

$$7x^4 = \frac{1}{5}(8P_4(x) + 30x^2 - 3) = \frac{8}{5}P_4(x) + 6x^2 - \frac{3}{5}.$$

By $P_2(x) = \frac{1}{2}(3x^2 - 1)$, we have

$$3x^2 = 2P_2(x) + 1 \Rightarrow 6x^2 = 4P_2(x) + 2.$$

Thus

$$7x^4 = \frac{8}{5}P_4(x) + 4P_2(x) + 2 - \frac{3}{5}, \Rightarrow c_4 = \frac{8}{5}, c_2 = 4, c_0 = 2 - \frac{3}{5} = 1.4.$$

4. Let $f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in [0, \pi). \end{cases}$ and let $f(x)$ be 2π-periodic. The Fourier series of $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right].$$

Find b_2.

(a) $\frac{2}{\pi^2}$ (b) $\frac{1}{\pi^2}$ (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$ (e) 0

Solution: (e).

$$\begin{aligned} b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(2x)dx + \int_{0}^{\pi} 1\sin(2x)dx\right) = -\frac{1}{2\pi}\cos(2x)|_0^{\pi} \\ &= -\frac{\cos(2\pi) - 1}{2\pi} = 0. \end{aligned}$$

Thus, $b_2 = 0$.

5. Let $f(x) = 1 - x$, $0 < x < 2$. Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$ be the Fourier cosine series of $f(x)$. Find a_2.

a) $\frac{2}{\pi^2}$ (b) $\frac{1}{\pi^2}$ (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$ (e) 0

Solution: (e).

$$\begin{aligned} a_2 &= \frac{2}{L}\int_0^L f(x)\cos(\frac{2\pi x}{L})\,dx = \int_0^2 (1-x)\cos(\pi x)\,dx \\ &= \left[\frac{1}{\pi}(1-x)\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x)\right]_0^2 \\ &= -\frac{1}{\pi^2}[\cos(2\pi) - \cos 0] = 0. \end{aligned}$$

6. [**4 points**] Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the series solution of the differential equation

$$y'' + xy' + y = 0$$

about $x = 0$. Find the coefficient recursive relation, i.e., the relation between a_{n+2} and a_n.

Solution: From $y = \sum_{n=0}^{\infty} a_n x^n$ we have

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute them into the differential equation,

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

i.e.,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$

i.e.,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_n x^n = 0,$$

i.e.,

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+1) a_n] x^n = 0.$$

Thus

$$a_{n+2} = -\frac{(n+1)a_n}{(n+2)(n+1)} = -\frac{a_n}{n+2}.$$

7. [**6 points**] Consider the following differential equation:

$$2xy'' + y' - y = 0, \quad x > 0.$$

(i) (2 points) Construct and solve the indicial equation.

(ii) (4 points) Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ be a solution of the differential equation. Find the recursive relation for c_n.

Solution: (i) We rewrite the DE as

$$y'' + \frac{1}{2x}y' - \frac{1}{x}y = 0.$$

Then

$$xp(x) = 0.5, \quad x^2q(x) = -x.$$

Thus $p_0 = 0.5, q_0 = 0$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 - 0.5r = 0. \Rightarrow r_1 = 0.5, r_2 = 0.$$

(ii) From

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

Substitute them into the differential equation, we have

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0. \Rightarrow$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0, \Rightarrow$$

$$r(2r-1)c_0 x^{r-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0, \Rightarrow$$

$$r(2r-1)c_0 x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(2n+2r+1)c_{n+1} x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0, \Rightarrow$$

$$r(2r-1)c_0 x^{r-1} + \sum_{n=0}^{\infty}[(n+r+1)(2n+2r+1)c_{n+1} - c_n]x^{n+r} = 0, \Rightarrow$$

$$c_{n+1} = \frac{c_n}{(n+r+1)(2n+2r+1)}, \quad n \geq 0.$$

8. [**5 points**] Consider the Sturm-Liouville equation:

$$y'' + \lambda y = 0, \quad y'(0) = 0, y(\pi) = 0.$$

Assume that $\lambda > 0$. Find the eigenvalues and the eigenfunctions.

Solution: Since $\lambda > 0$. Let $\lambda = \nu^2$, $\nu > 0$. Then we have

$$y'' + \nu^2 y = 0.$$

The solution is

$$y = c\cos\nu x + d\sin\nu x, \Rightarrow y' = -c\nu\sin\nu x + d\nu\cos\nu x.$$

Now we consider the two initial conditions:

$$y'(0) = 0, d\nu = 0, \Rightarrow d = 0, \Rightarrow y = c\cos\nu x;$$

$$y(\pi) = 0, \Rightarrow c\cos\nu\pi = 0,$$

and this can be satisfied with $c \neq 0$ provided that

$$\cos\nu\pi = 0, \Rightarrow \nu\pi = (n + \frac{1}{2})\pi, \nu = \frac{2n+1}{2}, \quad n = 0, 1, 2, ...$$

Take $c = 1$. Then

$$y_n(x) = \cos\left(\frac{2n+1}{2}x\right).$$

Hence the eigenfunctions are

$$y(x) = \cos\left(\frac{2n+1}{2}x\right), \quad n = 0, 1, 2, ...$$

and the eigenvalues are

$$\lambda = \left(\frac{2n+1}{2}\right)^2, \quad n = 0, 1, 2, ...$$

MAT3320 Midterm Formula

Legendre's equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$
Legendre polynomial: For $-1 < x < 1$,
$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x),$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$
Bessel's equation: $x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0.$
Sturm-Liouville equation: $[r(x)y']' + [q(x) + \lambda p(x)]y = 0.$