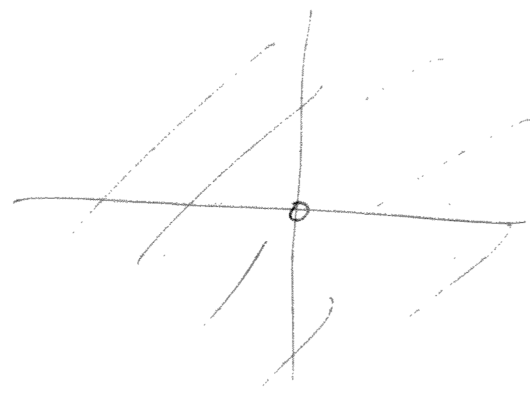
MAT 2122 HW #2

Solutions

1. The only way a point (x, y) won't be in the set is if the condition "$x \neq 0$ or $y \neq 0$" is not satisfied, that is, if "$x = 0$ and $y = 0$", so the set is $\mathbb{R}^2 \setminus \{(0,0)\}$.



To prove that $A = \mathbb{R}^2 \setminus \{(0,0)\}$ is open, we need, for any $\vec{x}_0 = (x_0, y_0) \in A$, a number $r > 0$ so that $D_r(\vec{x}_0) \subset A$.

So, take a particular $\vec{x}_0 = (x_0, y_0)$.

The ball $D_r(\vec{x}_0)$ is an open disc centred at $\vec{x}_0$ of radius r; this will be contained in A as long as it doesn't contain the point $(0,0)$.

One way to achieve this is to have $(0,0)$ on the boundary of the open disc, i.e. on the circle of radius r centred at (x_0, y_0). This property will define r: Given (x_0, y_0), there is only one radius r so that the circle centred at (x_0, y_0) of radius r passes through $(0,0)$; namely r = distance from (x_0, y_0) to $(0,0)$ $= \sqrt{x_0^2 + y_0^2} = \|\vec{x}_0\|$.

Since $\vec{x}_0 \in A$, we know that $x_0 \neq 0$ or $y_0 \neq 0$, and so

$r = \|\vec{x}_0\| > 0$.

So we have now chosen an $r > 0$ (namely $r = \|\vec{x}_0\|$),

such that $D_r(\vec{x}_0) \subset A$.

[proof, repeated: $A = \mathbb{R}^2 \setminus \{(0,0)\}$, so as long as $(0,0) \notin D_r(\vec{x}_0)$, then $D_r(\vec{x}_0) \subseteq A$. However $\|\vec{0} - \vec{x}_0\| = \|\vec{x}_0\| = r$ is not less than r, as would be required for $(0,0)$ to be in $D_r(\vec{x}_0)$; and hence $\vec{0} = (0,0) \notin D_r(\vec{x}_0)$.]

Hence A is open.

2 (a) Let's simplify. $(x+y)^2 - (x-y)^2 = (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)$

$= 4xy$,

so the limit is

$$\lim_{(x,y)\to(0,0)} \frac{4\cancel{xy}}{\cancel{xy}} = \lim_{(x,y)\to(0,0)} 4 = 4.$$

(b) Along the line $y=0$, this becomes $\frac{(x-0)^2}{x^2+0^2} = 1$.

Along the line $y=x$, this becomes $\frac{(x-x)^2}{x^2+x^2} = 0$.

So depending on how you approach $(0,0)$, the function approaches different values, and hence there is no single limit.

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$ does not exist.

2(c) This problem is trickier, and is treated as a bonus problem.

Along any line $y = kx$ through the origin $(0,0)$,

the expression becomes

$$\frac{x^3 - (kx)^3}{x \cdot (kx)} = \frac{1-k^3}{k} \cdot x ,$$

which approaches 0 as x approaches 0.

So we might guess that the limit is 0, and try to use our tricks to prove it.

However the x and y in the denominator cause problems - we know $|x| \leq \|(x,y)\|$, but how can you bound

$$\frac{1}{|x|} \text{ in terms of } \|(x,y)\| ?$$

You can't.

If you approach $(0,0)$ along the curve $y = x^2$ (instead of the lines considered above), then the expression becomes

$$\frac{x^3 - (x^2)^3}{x\,(x^2)} = \frac{x^3 - x^6}{x^3} = 1 - x^3 ,$$

which approaches 1 as $x \to 0$.

So, again, depending on how you approach $(0,0)$, the function approaches different values, and

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{xy} \text{ does not exist.}$$

3) The domain of f is $\mathbb{R}^2 \setminus \{(0,0)\}$; it's defined everywhere except where there is division by 0, which is at $(0,0)$.

$$\frac{\partial f}{\partial x} = \frac{y\left(\sqrt{x^2+y^2}\right) - (xy)\left(\cancel{\tfrac{1}{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x\right)}{x^2+y^2}$$

(quotient and chain rule in 1 dimension)

$$= \frac{y(\cancel{x^2}+y^2) - \cancel{(xy)(x)}}{(x^2+y^2)^{3/2}}$$

(taking $\sqrt{x^2+y^2}$ to the denominator)

$$= \frac{y^3}{(x^2+y^2)^{3/2}}.$$

Similarly, $$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

Both functions are built out of our basic functions, hence are continuous whenever they are defined; and they are both defined on $R^2 \setminus \{(0,0)\}$.

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are defined on the domain of f.

(Compare that to $f(x) = \sqrt[4]{x^2} = \sqrt{|x|}$, whose domain is $\mathbb{R}$, and whose derivative is not even defined on $\mathbb{R}$.)

4) $$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

(Each row is the different partial derivatives of a component of f.)

$$= \begin{bmatrix} e^x & 0 \\ y\cos(xy) & x\cos(xy) \end{bmatrix}$$

5) Their tangent planes at $(x,y) = (0,0)$ are given by :

- for $f(x,y)$, its linear approximation

$$z = f(0,0) + \nabla f(0,0) \cdot ((x,y) - (0,0)).$$

(where $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 2y)$),

$$z = 0 + (2\cdot 0, 2\cdot 0) \cdot ((x,y))$$

$$= 0 + (0,0) \cdot (x,y)$$

$$= 0.$$

Tangent plane of f at $(0,0)$ is $z = 0$.

- for $g(x,y)$, its linear approximation

$$z = g(0,0) + \nabla g(0,0) \cdot ((x,y) - (0,0))$$

$$= 0$$

(here $\nabla g(x,y) = (-2x+y^3,\ -2y+3xy^2)$, and so

$$\nabla g(0,0) = (-2\cdot 0 + 0^3,\ -2\cdot 0 + 3\cdot 0\cdot 0^2) = (0,0))$$

Since their tangent planes match, we can say the graphs are tangent to each other at $(0,0)$.

6) The tangent line to $c(t)$ at $t_0=0$ is given by its linear approx:

$$L(t) = c(t_0) + c'(t_0)(t-t_0)$$

Here $t_0=0$, $c(t_0) = c(0) = (1,0,0)$,

$c'(t) = (2\cos t\,(-\sin t),\ 3-3t^2,\ 1)$, so

$c'(t_0) = c'(0) = (0,3,1)$, and so

$$L(t) = (1,0,0) + (0,3,1)(t).$$
$$= (1, 3t, t).$$

Again, $\boxed{L(t) = (1,3t,t)}$

(7) The chain rule for $\frac{\partial h}{\partial x}$ says that (*) $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}$,

since if we write $g(x,y) = (u(x,y), v(x,y))$, $h = f\circ g$, and so

$$\begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y}\end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix},$$

(i.e. $Dh = Df\cdot Dg$)

at least, when evaluated at the correct points, — multiplying out the matrices gives the identity (*).

Let's compute both sides of (*).

$$h(x,y) = \frac{u^2+v^2}{u^2-v^2} = \frac{(e^{-x-y})^2 + (e^{xy})^2}{(e^{-x-y})^2 - (e^{xy})^2} = \frac{e^{-2x-2y} + e^{2xy}}{e^{-2x-2y} - e^{2xy}}.$$

$$\frac{\partial h}{\partial x} = \frac{(-2e^{-2x-2y} + 2ye^{xy})(e^{-2x-2y} - e^{2xy}) - (e^{-2x-2y} + e^{2xy})(-2e^{-2x-2y} - 2ye^{2xy})}{(e^{-2x-2y} - e^{2xy})^2}$$

We won't need to simplify. — the arrows are for later use.

Now

$$\frac{\partial f}{\partial u} = \frac{2u(u^2-v^2) - (u^2+v^2)(2u)}{(u^2-v^2)^2},$$

$$\frac{\partial u}{\partial x} = -e^{-x-y},$$

and their product is, substituting in the formulas for u and v in some places

$$\frac{2e^{-x-y}(u^2-v^2) - (u^2+v^2)2e^{-x-y}}{(u^2-v^2)^2}\left(-e^{-x-y}\right),$$

which corresponds to the terms in $\frac{\partial h}{\partial x}$ with the arrows.

The other terms in $\frac{\partial h}{\partial x}$ will similarly come from $\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$.

8) $\nabla f(x,y) = (1+2y,\ 2x-6y)$, so $\nabla f(1,2) = (5,-10)$.

The directional derivative is $\nabla f(1,2)\cdot\vec{v} = (5,-10)\cdot(\frac{3}{5},\frac{4}{5})$

$$= 3 - 8 = -5.$$

9. The tangent plane to $f(\vec{x}) = c$ at $\vec{x}_0$ is

$$\nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) = 0.$$

Here $\vec{x}_0 = (1,2,\frac{1}{3})$.

$f(x,y,z) = x^2 + 2y^2 + 3xz$, $\quad c = 10$. ← must be a constant!

$\nabla f = (2x+3z,\ 4y,\ 3x)$, so

$\nabla f(\vec{x}_0) = (3,8,3)$, and the equation is, writing (x,y,z) for $\vec{x}$,

$$(3,8,3)\cdot(x-1,\ y-2,\ z-\tfrac{1}{3}) = 0,$$

$$3(x-1) + 8(y-2) + 3(z-\tfrac{1}{3}) = 0$$

$$3x + 8y + 3z = 3 + 16 + 1 = 20.$$