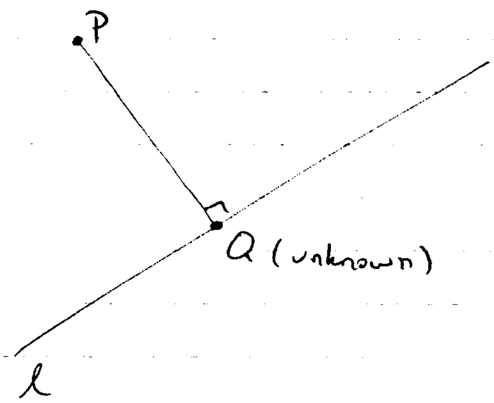
MAT 2122 HW#1
Solutions

1.2 #18. As suggested by the hint, we assign coordinates to the intersection point; call it $Q=(x_0, y_0, z_0)$.

Also write $P = (3,1,-2)$.



Then we require that Q is on the line, and also that the vector $\overrightarrow{PQ}$ is orthogonal to the line.

Q on the line means that for some $t_0 \in \mathbb{R}$,

$$x_0 = -1 + t_0$$
$$y_0 = -2 + t_0$$
$$z_0 = -1 + t_0.$$

$\overrightarrow{PQ}$ orthogonal to the line means that $\overrightarrow{PQ}$ is orthogonal to the direction vector of the line, which is $(1,1,1)$.

[The line is $\{(-1,-2,-1) + (1,1,1)t \mid t \in \mathbb{R}\}$, so its direction vector is $(1,1,1)$.]

We test this by setting the dot product to be 0.

$$(x_0 - 3,\ y_0 - 1,\ z_0 - 2) \cdot (1,1,1) = 0$$

$$(x_0 - 3)\cdot 1 + (y_0 - 1)\cdot 1 + (z_0 + 2)\cdot 1 = 0$$

$$x_0 + y_0 + z_0 = 2$$

Substituting in the formulas for x_0, y_0, z_0 in terms of t_0, we get

$$(-1 + t_0) + (-2 + t_0) + (-1 + t_0) = 2$$
$$3t_0 = 6 \quad , \quad t_0 = 2.$$

So the point Q is $(-1+2, -2+2, -1+2) = (1, 0, 1)$.

The line through P and Q is the desired line. It is given by

$$\{(3,1,-2) + (1-3,\ 0-1,\ 1+2)t \mid t \in \mathbb{R}\}$$

$$= \{(3,1,-2) + (-2,-1,3)t \mid t \in \mathbb{R}\}$$

Notice that $t = 0, 1$, corresponds to the points P, Q, respectively.

1.3 #10 First we construct $\underline{a}$ vector orthogonal to the two vectors, using the cross product:

$$\vec{V} = \begin{vmatrix} \vec{i} & -5 & 7 \\ \vec{j} & 9 & 8 \\ \vec{k} & -4 & 9 \end{vmatrix} = (9\cdot 9 + 4\cdot 8)\vec{i} - (-5\cdot 9 + 4\cdot 7)\vec{j} + (-5\cdot 8 - 9\cdot 7)\vec{k}$$

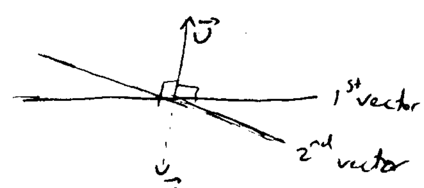
$$= 113\vec{i} + 17\vec{j} - 103\vec{k}.$$

Next we rescale it to be a unit vector:

$$\vec{U} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{113\vec{i} + 17\vec{j} - 113\vec{k}}{\sqrt{(113)^2 + 17^2 + (-103)^2}} = \frac{113\vec{i} + 17\vec{j} - 113\vec{k}}{\sqrt{23667}}.$$

The vector $\vec{U}$ is a unit vector orthogonal to both vectors.

There is only one other vector satisfying these properties: $-\vec{U}$.



So the answer is $\vec{U}$ and $-\vec{U}$, with $\vec{U}$ as above.

1.3 #16 (c) Write $P = (2, -1, 3)$, $Q = (0, 0, 5)$, $R = (5, 7, -1)$.

Then the desired plane passes through P and is parallel to the vectors $\vec{PQ}$ and $\vec{PR}$, so one description of the plane is

$$\{ P + s\vec{PQ} + t\vec{PR} \mid s, t \in \mathbb{R} \}.$$

This isn't an equation, though. We get an equation using P and a vector orthogonal to the plane (a <u>normal</u> to the plane).

One vector that is normal is

$$\vec{n} = \vec{PQ} \times \vec{PR} = (0-2,\ 0-(-1),\ 5-3) \times (5-2,\ 7-(-1),\ -1-3)$$

$$= (-2, 1, 2) \times (3, 8, -4)$$

$$= \begin{vmatrix} \vec{i} & -2 & 3 \\ \vec{j} & 1 & 8 \\ \vec{k} & 2 & -4 \end{vmatrix}$$

(expand along 1st column)

$$= (-4-16)\vec{i} - (8-6)\vec{j} + (-16-3)\vec{k}$$

$$= -20\vec{i} - 2\vec{j} - 19\vec{k} = (-20, -2, -19).$$

Given P and $\vec{n}$, the equation for the plane is

$$\vec{n} \cdot (\vec{x} - P) = 0,$$

i.e, if $\vec{x} = (x, y, z)$, then $\vec{x}$ is in the plane exactly when

$$\vec{n} \cdot (\vec{x} - P) = (-20, -2, -19) \cdot (x-2,\ y+1,\ z-3)$$

$$= -20(x-2) - 2(y+1) - 19(z-3)$$

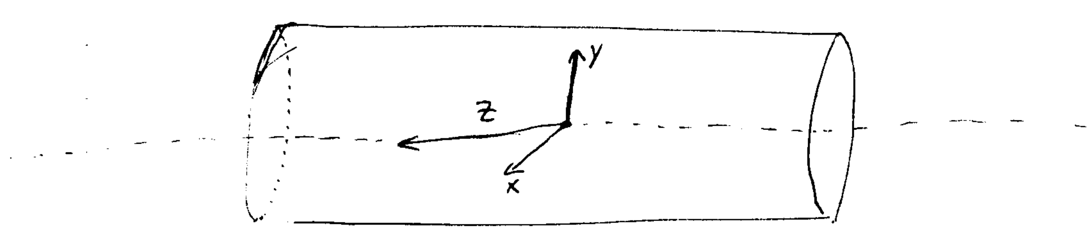
$$= -20x - 2y - 19z + 95 \qquad \text{equals } 0,$$

or $\quad 20x + 2y + 19z = 95.$

1.4 #12 The first step is choose our coordinate axes in a suitable way relative to the tank. We'll take our units as ft.

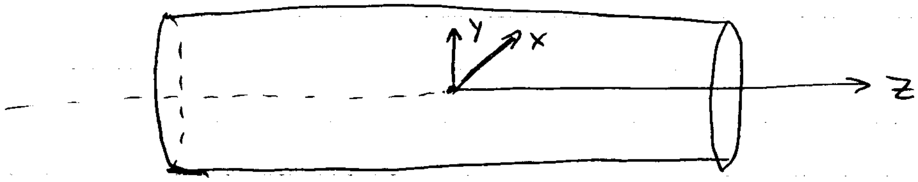
To have the answer come out at all reasonable, we need to choose the z-axis to lie horizontally along the centre axis of the tank.

We will choose the origin to be the exact centre of the tank, but other choices are also correct. We will also choose the x-axis to be horizontal and the y-axis to point straight up.



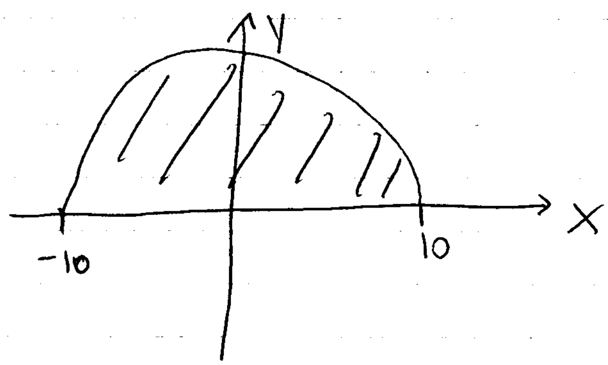
For a given cylinder, there is still another choice to be made; there are two sets of axes satisfying all the conditions above.

The other is



Luckily, the description of the air space is the same with both these choices.

The z value of any point in the tank must be between -8 and 8 (total length 16), and the cross section of air space looks the same for each z in this interval. Each cross section looks like



the points in the disc of radius 10 centred at (0,0), with $y > 0$.

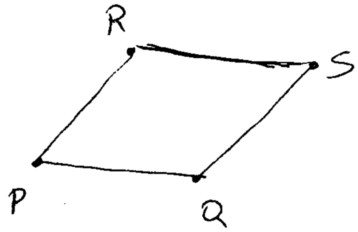
In terms of r and θ, this is
$$0 \leq r \leq 10$$
$$0 \leq \theta \leq \pi.$$

So the air space is described by the set

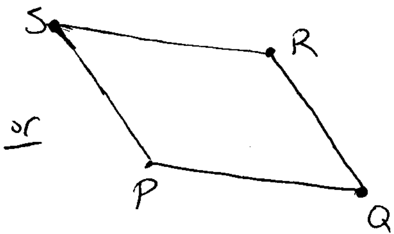
$$\{(r, \theta, z)_{cyl} \mid 0 \leq r \leq 10,\ 0 \leq \theta \leq \pi,\ -8 \leq z \leq 8\}.$$

1. Review #14. This problem is not correctly worded, as there is more than one parallelepiped described by these 4 points.

Similarly, 3 points describe more than one parallelogram: Given P, Q, R, we can build the 4th corner S as follows.



, or



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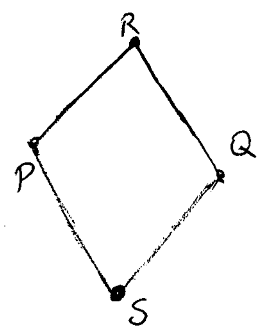
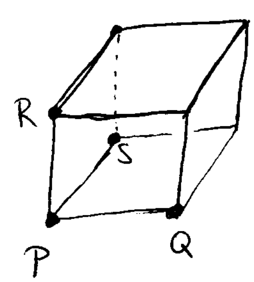


. However they will all turn out to have the same area (by properties of determinants).

Let's call the 4 given points P, Q, R, S, in order, and build the parallelepiped as below.



It has P as one corner, and is spanned by $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$.

Its volume is the determinant $\left| \vec{PQ} \quad \vec{PR} \quad \vec{PS} \right|$

absolute value of the

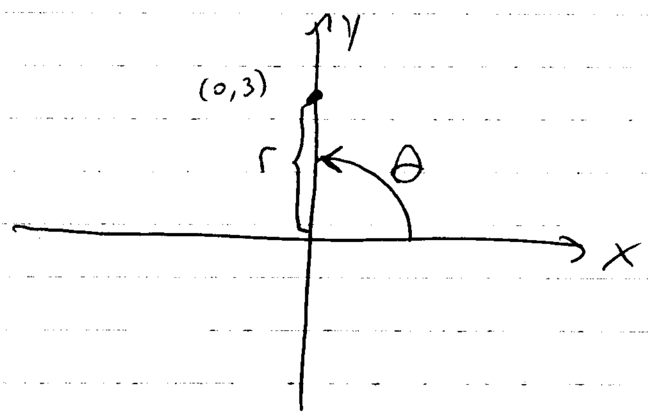
The determinant is (since $\vec{PQ} = (1,1,1) - (0,1,0) = (1,0,1)$, etc.)

$$\begin{array}{c} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{PQ} & \vec{PR} & \vec{PS} \end{array} \end{array} = 1(2) + 0 + 3(-1) = -1,$$

expanding along 1^{st} row.

So the volume is $|-1| = 1$.

1. Review 22(a) We have $x=0$, $y=3$, $z=4$. For cylindrical coords, plot (x,y) in the xy plane.



r = distance from origin $= 3$.
θ = angle, in counterclockwise direction from positive x-axis
$= \pi/2$.

Also, $z=4$, so the point is $(3, \pi/2, 4)_{cyl}$.

For spherical coordinates, $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3^2+0^2+4^2} = 5$,

and $\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{4}{5}\right) \sim 0.6435$

θ is the same as in cylindrical, so the point is $(4, \pi/2, \cos^{-1}(\frac{4}{5}))_{Sph}$
See plot next page

1. Review

24 (a) We're given $\rho = 1$, $\theta = \pi/2$, $\phi = \pi$.

We have formulas to get x, y, z:

$$x = \rho \cos\theta \sin\phi = 1 \cdot \cos\tfrac{\pi}{2} \cdot \sin(\pi) = 0$$
$$y = \rho \sin\theta \sin\phi = 1 \cdot \sin(\pi/2) \cdot \sin(\pi) = 0$$
$$z = \rho \cos\phi = 1 \cdot \cos(\pi) = -1,$$

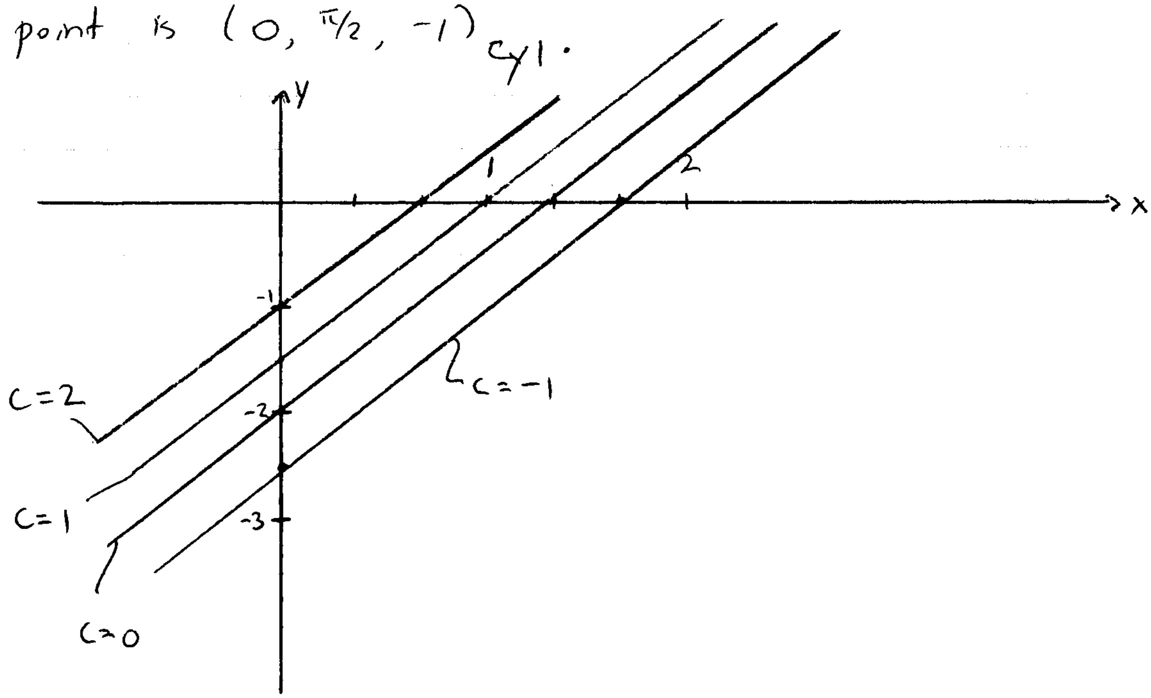
So the point is $(0,0,-1)$. See plot next page.

In cylindrical coordinates $r = \sqrt{x^2+y^2} = 0$,

We can pick any value for θ, but the choice that makes the most sense is the θ from the spherical coordinates $\theta = \pi/2$.

So the point is $(0, \pi/2, -1)_{cyl}$.

2.1 #4



$c=-1$ is the line $f(x,y) = -1$; $4 - 3x + 2y = -1$, $y = \frac{3}{2}x - \frac{5}{2}$
etc. The graph of f is a plane.

Bonus Let $A = D_r(\vec{x}_0) \setminus \{\vec{x}_0\} = \{\vec{x} \in \mathbb{R}^n \mid 0 < \|\vec{x} - \vec{x}_0\| < r\}$.

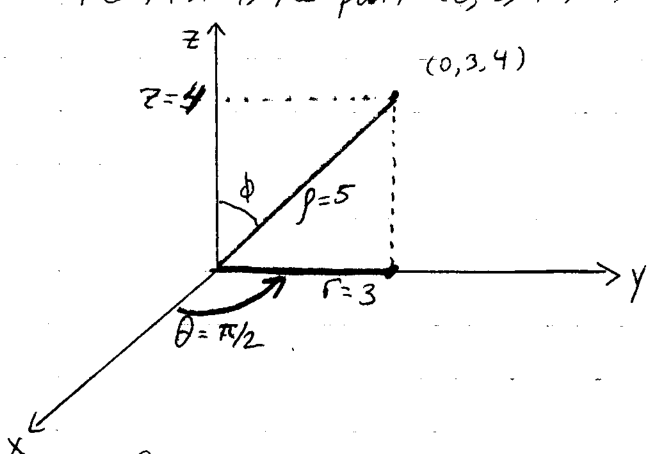
To prove A is open, we need to show that for every $\vec{a} \in A$, there is a nbhd of a inside A. If we choose as a nbhd $D_s(\vec{a})$, then we simply have to find an appropriate $s > 0$.

One that works is $s = \min(\|\vec{x} - \vec{x}_0\|, r - \|\vec{x} - \vec{x}_0\|)$.

Details will be provided in Friday's DGD.

Plot of the points in 1. Review #22 and 24:

The first is the point $(0,3,4)$: its projection to the xy plane is on the positive y-axis.



The second is $(0,0,-1)$; it's on the negative z-axis