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EMAT 213/4 FINAL EXAM MAY 2, 2002

Instructions: Answer all questions. Calculators allowed.

1. (8 pts) Solve $\frac{dy}{dx} = 6xy$.
2. (10 pts) Solve the intial value problem $2xy' + y = 2$, $y(4) = 4$.
3. (8 pts) Solve $4xydx + 2(x^2 - 1)dy = 0$.
4. (8 pts) Solve by using a substitution: $(y^2 + xy)dx - x^2dy = 0$.
5. (12 pts) When a quiche is removed from an oven, its temperature is measured at 180°C. The room temperature is 20°C. Ten minutes later, the temperature of the quiche is measured at 140°C. How long after it is taken out of the oven will the quiche's temperature be 120°C ?
6. (10 pts) Solve the initial value problem $y'' - 5y' + 6y = 0$, $y(1) = 0$, $y'(1) = 4$.
7. (12 pts) Solve $y'' - 4y' + 4y = e^{2x} \cos x$.
8. (10 pts) Solve by variation of parameters: $y'' + y = \sin^2 x$.
9. (10 pts) A force of 200 newtons stretches a spring 2 meters. A mass of 30 kilograms is attached to the end of the spring and released from the equilibrium position with an upward velocity of 5 meters/sec. Find the equation of motion.
10. (12 pts) Use diagonalization to solve

$$X' = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} X + \begin{pmatrix} t \\ 3 \end{pmatrix}.$$

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Solutions to final Exam May 2002

no1 Solve $\frac{dy}{dx} = 6xy$

Solution This is a separable DE of order 1

$$\frac{dy}{dx} = 6xy \iff \frac{dy}{y} = 6x\,dx \iff \int \frac{dy}{y} = \int 6x\,dx$$

$$\iff \ln(y) = 3x^2 + C \iff y = e^{3x^2+C}$$

$$\iff y = Ce^{3x^2} \text{ renaming } C.$$

no2 Solve the initial value problem

$$2xy' + y = 2, \quad y(4) = 4$$

solution This is a linear DE of order 1. Under standard form: $y' + \frac{y}{2x} = \frac{1}{x}$

$$p(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\int \frac{1}{x}dx} = e^{\frac{1}{2}\ln x} = e^{\ln x^{1/2}} = x^{1/2}$$

Then, $y(x) = \frac{1}{x^{1/2}} \int x^{1/2} \frac{1}{x}\,dx$

$$= \frac{1}{x^{1/2}} \int x^{-1/2}\,dx = \frac{1}{x^{1/2}} \left(\frac{x^{1/2}}{1/2} + C \right)$$

$$\implies y(x) = 2 + Cx^{-1/2}$$

With $y(4) = 4$ $\quad 4 = 2 + C\,4^{-1/2}$

$$\iff C = 2 \cdot 2 = 4$$

Particular solution $y(x) = 2 + 4x^{-1/2}$.

no 3. Solve $4xy\,dx + 2(x^2-1)\,dy = 0$

solution $\frac{\partial M}{\partial y} = 4x$ and $\frac{\partial N}{\partial x} = 2(2x) = 4x$

$\Rightarrow$ exact DE

We solve ① $\frac{\partial f}{\partial x} = 4xy$

② $\frac{\partial f}{\partial y} = 2x^2 - 2$

Then, $f(x,y) = \int 4xy\,dx = 2x^2y + C(y)$

$\Rightarrow \frac{\partial f}{\partial y} = 2x^2 + C'(y) = ② = 2x^2 - 2$

$\Rightarrow C'(y) = -2 \Rightarrow C(y) = -2y + C$

Then, $f(x,y) = 2x^2y - 2y + C$ satisfy ① & ②, and the general solution of the DE is

$$2x^2y - 2y = C.$$

$\boxed{*\ \text{no 4 is at the end}}$

no 5

solution Recall Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_s) = k(T-20)$$

as T_s = room temperature = 20

~~This is a separable DE~~

$$\frac{dT}{dt} = k(T-20) \Leftrightarrow \int \frac{dT}{T-20} = \int k\,dt$$

$\Leftrightarrow \ln(T-20) = kt + C$

$\Leftrightarrow T - 20 = e^{kt+C} = Ce^{kt}$ renaming C

$\Leftrightarrow T(t) = 20 + Ce^{kt}$

To find C $\quad T(0) = 180$

$$180 = 20 + Ce^{0} \Rightarrow C = 160$$

To find k $\quad T(10) = 140$

$$140 = 20 + 160\,e^{10k}$$

$$\Leftrightarrow \frac{120}{160} = e^{10k} \Leftrightarrow 10k = \ln\left(3/4\right)$$

$$\Leftrightarrow k = \frac{1}{10}\ln\left(3/4\right) \simeq -0.0287$$

What is t st $T(t) = 120$?

$$120 = 20 + 160\,e^{kt}$$

$$\Leftrightarrow e^{kt} = \frac{100}{160} = \frac{5}{8}$$

$$\Leftrightarrow t = \frac{1}{k}\ln\left(5/8\right) \simeq 16.33 \text{ min.}$$

no6 Solve the IVP $y'' - 5y' + 6y = 0$, $y(1) = 0$
$y'(1) = 4$

solution The auxiliary equation is

$$m^2 - 5m + 6 = (m-2)(m-3) = 0$$

$$\Rightarrow m_1 = 2 \text{ and } m_2 = 3$$

general solution $\quad y(x) = C_1 e^{2x} + C_2 e^{3x}$

$$\Rightarrow y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$$

$y(1) = 0$ $\quad 0 = C_1 e^{2} + C_2 e^{3}$ $\quad$ ①

$y'(1) = 4$ $\quad 4 = 2C_1 e^{2} + 3C_2 e^{3}$ $\quad$ ②

② − 2① gives $\quad 4 = C_2 e^3 \Rightarrow C_2 = \dfrac{4}{e^3} \simeq 0.1991$

$$\Rightarrow C_1 = -\frac{C_2 e^3}{e^2} = -C_2 e \simeq -0.5413$$

Then, the particular solution is

$$y(x) = -0.5413 e^{2x} + 0.1991 e^{3x} .$$

no 7 $\quad$ Solve $y'' - 4y' + 4y = e^{2x}\cos x$

solution $\quad$ We first solve the homogeneous Eq.

The auxiliary Eq is

$$m^2 - 4m + 4 = (m-2)^2 = 0$$

ie $m = 2$ is a double root

Then $y_c(x) = C_1 e^{2x} + C_2 x e^{2x}$

To find $y_p(x)$, we use underdetermined coefficients

$$y_p(x) = A e^{2x}\cos x + B e^{2x}\sin x$$

$$\begin{aligned}
y_p'(x) &= 2Ae^{2x}\cos x - Ae^{2x}\sin x \\
&\quad + 2Be^{2x}\sin x + Be^{2x}\cos x \\
&= (2A+B)e^{2x}\cos x + (2B-A)e^{2x}\sin x
\end{aligned}$$

$$\begin{aligned}
y_p''(x) &= 2(2A+B)e^{2x}\cos x - (2A+B)e^{2x}\sin x \\
&\quad + 2(2B-A)e^{2x}\sin x + (2B-A)e^{2x}\cos x \\
&= e^{2x}\cos x\,(4A+2B+2B-A) \\
&\quad + e^{2x}\sin x\,(-2A-B+4B-2A) \\
&= (3A+4B)e^{2x}\cos x + (3B-4A)e^{2x}\sin x
\end{aligned}$$

$\longrightarrow$

Then, $y_p'' - 4y_p' + 4y_p =$

$(3A+4B)e^{2x}\cos x + (3B-4A)e^{2x}\sin x$

$-4(2A+B)e^{2x}\cos x - 4(2B-A)e^{2x}\sin x$

$+4Ae^{2x}\cos x + 4Be^{2x}\sin x$

$= e^{2x}\cos x\,(3A+4B-8A-4B+4A)$

$+ e^{2x}\sin x\,(3B-4A-8A+4A+4B)$

$= -Ae^{2x}\cos x - Be^{2x}\sin x$

which should be $e^{2x}\cos x$

$\Rightarrow A=-1$ and $B=0$

Then, $y_p(x) = -e^{2x}\cos x$

and $y(x) = y_c(x) + y_p(x)$

$= C_1e^{2x} + C_2xe^{2x} - e^{2x}\cos x.$

no 8 Solve by variation of parameters

$$y'' + y = \sin^2 x$$

solution The auxiliary Eq is $m^2+1=0$

$\Leftrightarrow m = \pm i$

Then, $y_c(x) = C_1\cos x + C_2\sin x$.

Using variation of parameters, we set

$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

with $y_1(x) = \cos x$

$y_2(x) = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \sin^2 x & \cos x \end{vmatrix}}{W} = -\sin^3 x$$

$$u_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sin^2 x \end{vmatrix}}{W} = \cos x \sin^2 x$$

Then $u_1(x) = -\int \sin^3 x \, dx = -\int \sin x (\sin^2 x) \, dx$

$$= -\int \sin x (1 - \cos^2 x) \, dx$$

$$= -\int \sin x \, dx + \int \sin x \cos^2 x \, dx$$

$$= \cos x - \frac{\cos^3 x}{3}$$

$$u_2(x) = \int \cos x \sin^2 x \, dx = \frac{\sin^3 x}{3}$$

Then, $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$

$$= \left(\cos x - \frac{\cos^3 x}{3}\right) \cos x + \left(\frac{\sin^3 x}{3}\right) \sin x$$

$$= \cos^2 x - \frac{\cos^4 x}{3} + \frac{\sin^4 x}{3}$$

and $y(x) = y_c(x) + y_p(x)$

$$= C_1 \cos x + C_2 \sin x + \cos^2 x - \frac{\cos^4 x}{3} + \frac{\sin^4 x}{3}$$

no 9.

solution $m = 30\,kg$, $k = \frac{\text{weight}}{\text{displacement}} = \frac{200}{2} = 100$

DE $mx'' + kx = 0$

$30x'' + 100x = 0$

auxiliary equation $30m^2 + 100 = 0$

roots $\frac{\pm\sqrt{-120 \cdot 100}}{60}$

$= \frac{\pm 20\sqrt{30}\,i}{60} = \pm\frac{\sqrt{30}}{3}i$

Equation of motion

$x(t) = C_1 \cos\left(\frac{\sqrt{30}}{3}t\right) + C_2 \sin\left(\frac{\sqrt{30}}{3}t\right)$

Initial conditions $x(0) = 0$

$x'(0) = -5$

$x(0) = 0$ $\quad 0 = C_1\cos(0) + C_2\sin(0) \Rightarrow C_1 = 0$

$x(t) = C_2 \sin\left(\frac{\sqrt{30}}{3}t\right)$

$x'(t) = \frac{\sqrt{30}}{3} C_2 \cos\left(\frac{\sqrt{30}}{3}t\right)$

$x'(0) = -5$ $\quad -5 = \frac{\sqrt{30}}{3} C_2 \cos(0)$

$\Rightarrow C_2 = \frac{-15}{\sqrt{30}} = \frac{-15\sqrt{30}}{30} = \frac{-\sqrt{30}}{2}$

Then, $x(t) = -\frac{\sqrt{30}}{2}\sin\left(\frac{\sqrt{30}}{3}t\right)$.

no10 - Use diagonalisation to solve $\overline{X}' = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \overline{X} + \begin{pmatrix} t \\ 3 \end{pmatrix}$

solution We first find the eigenvalues & eigenvectors of $A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} = (\lambda - 3)^2 - 9$$

$$= \lambda^2 - 6\lambda + 9 - 9 = \lambda^2 - 6\lambda = \lambda(\lambda - 6)$$

Eigenvalues $\lambda_1 = 0$ & $\lambda_2 = 6$

Eigenvectors for $\lambda_1 = 0$

$$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

k_2 free, $k_1 = -k_2$

$\vec{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda_2 = 6$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

k_2 free, $k_1 = k_2$

$\vec{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $P = \begin{pmatrix} \vec{K}_1 & \vec{K}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

is such that $P^{-1} A P = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$

Setting $\overline{X} = P\overline{Y}$, we have the system

$$\overline{Y}' = D\overline{Y} + P^{-1}\vec{F} \quad \text{with } P^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} t \\ 3 \end{pmatrix}$$

$\Leftrightarrow \quad y_1' = -\frac{1}{2}t + \frac{3}{2}$ ①

$y_2' = 6y_2 + \frac{1}{2}t + \frac{3}{2}$ ②

<u>solution of ①</u> $y_1 = \int -\frac{1}{2}t + \frac{3}{2} \, dt = -\frac{t^2}{4} + \frac{3}{2}t + C_1$

<u>solution of ②</u> $y_2' - 6y_2 = \frac{1}{2}t + \frac{3}{2}$ <u>Linear</u>

$$e^{\int -6\,dt} = e^{-6t}$$

$$\Rightarrow y_2(t) = e^{6t} \int \left(\frac{1}{2}t + \frac{3}{2}\right) e^{-6t} \, dt$$

$$= e^{6t} \left(-\frac{1}{12} t e^{-6t} - \frac{19}{72} e^{-6t} + C_2 \right)$$

integrating by part

Then, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{t^2}{4} + \frac{3}{2}t + C_1 \\ -\frac{1}{12}t - \frac{19}{72} + C_2 e^{6t} \end{pmatrix}$

and $\underline{X} = P\overline{Y} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{t^2}{4} + \frac{3}{2}t + C_1 \\ -\frac{1}{12}t - \frac{19}{72} + C_2 e^{6t} \end{pmatrix}$

$$= \begin{pmatrix} \frac{t^2}{4} - \frac{3}{2}t - C_1 - \frac{1}{12}t - \frac{19}{72} + C_2 e^{6t} \\ -\frac{t^2}{4} + \frac{3}{2}t + C_1 - \frac{1}{12}t - \frac{19}{72} + C_2 e^{6t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t^2}{4} - \frac{19}{12}t - C_1 - \frac{19}{72} + C_2 e^{6t} \\ -\frac{t^2}{4} + \frac{17}{12}t + C_1 - \frac{19}{72} + C_2 e^{6t} \end{pmatrix}$$

$$= C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2/4 + \begin{pmatrix} -19/12 \\ 17/12 \end{pmatrix} t + \begin{pmatrix} -19/72 \\ -19/72 \end{pmatrix}.$$

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no4 Solve by using a substitution

$$(y^2 + xy)\,dx - x^2\,dy = 0$$

solution $M(x,y)$ & $N(x,y)$ homogeneous of degree 2

Then use $y = ux$

$dy = u\,dx + x\,du$

which gives

$$(u^2x^2 + x\,ux)\,dx - x^2(u\,dx + x\,du) = 0$$

$\Leftrightarrow (u^2x^2 + \cancel{x^2u} - \cancel{x^2u})\,dx - x^3\,du = 0$

$\Leftrightarrow u^2x^2\,dx - x^3\,du = 0$

$\Leftrightarrow \dfrac{du}{dx} = \dfrac{u^2x^2}{x^3} = \dfrac{u^2}{x}$ separable

$\Leftrightarrow \displaystyle\int \frac{du}{u^2} = \int \frac{dx}{x} \Leftrightarrow -u^{-1} = \ln(x) + C$

and replacing $u = \frac{y}{x}$, we get

$$-\frac{x}{y} = \ln(x) + C.$$

Concordia
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Department of Mathematics & Statistics

Course	Number	Section(s)
Engineering Mathematics	213/2	All
Examination	**Date**	**Pages**
Final	December 2001	1

Instructors

M. Bertola, P. Bracken, J. Hayes, A. Keviczky, Y. Khidirov, A. Kokotov, D. Korotkin

Note:

- ▶ Answer all questions.
- ▶ Calculators are allowed.

1. (8 pts) Solve $\frac{dy}{dx} = -4xy$.

2. (10 pts) Solve the intial value problem $2xy' + y = e^{2x}$, $y(2) = 2$.

3. (8 pts) Solve $(y^2 - 1)dx + 2xydy = 0$.

4. (8 pts) Solve by using a substitution: $-y^2dx + (x^2 + xy)dy = 0$.

5. (12 pts) When a potato is removed from an oven, its temperature is measured at 200°C. Five minutes later, its temperature is measured at 150°C. How long after it is taken out of the oven will the potato's temperature be 100°C ?

6. (10 pts) Solve the initial value problem $y''-4y'-5y = 0$, $y(1) = 0$, $y'(1) = 2$.

7. (12 pts) Solve $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$.

8. (10 pts) Solve by variation of parameters: $y'' + 3y' + 2y = \frac{1}{1+e^x}$.

9. (10 pts) A force of 300 newtons stretches a spring 3 meters. A mass of 60 kilograms is attached to the end of the spring and released from the equilibrium position with an upward velocity of 5 meters/sec. Find the equation of motion.

10. (12 pts) Use diagonalization to solve

$$X' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} X + \begin{pmatrix} t \\ \frac{1}{2} \end{pmatrix}.$$

Concordia
UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Section(s)
Engineering Mathematics	213/2	All
Examination	**Date**	**Pages**
Final	December 2001	1

Instructors

M. Bertola. P. Bracken. J. Hayes. A. Keviczky. Y. Khidirov. A. Kokotov. D. Korotkin

Note:

- Answer all questions.
- Calculators are allowed.

1. (8 pts) Solve $\frac{dy}{dx} = -4xy$.
2. (10 pts) Solve the intial value problem $2xy' + y = e^{2x}$, $y(2) = 2$.
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10. (12 pts) Use diagonalization to solve

$$X' = \begin{pmatrix} \text{[illegible]} & \text{[illegible]} \\ \text{[illegible]} & \text{[illegible]} \end{pmatrix} X + \begin{pmatrix} \text{[illegible]} \\ \text{[illegible]} \end{pmatrix}.$$

1.20

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Concordia University

Final Examination - EMAT 213 - All sections

[redacted]

Date: December 2003
Time Allowed: 3 hours
Instructors: M. Bertola, C. David, D. Dryanov, J. Hayes, Q. Katatbeh, M. Mei, A. Pal
Course Examiner: C. David
Directions: Answer all questions. NO CALCULATORS.

MARKS

(9) **1.** Solve the differential equation

$$y' \cos^2 x \sin x + y \cos^3 x = \sin x.$$

(9) **2.** Solve the differential equation

$$(2xy^2 - 3)\, dx + (2x^2y + 4)\, dy = 0.$$

(9) **3.** Solve the differential equation

$$y^{1/2} \frac{dy}{dx} + y^{3/2} = 1.$$

(9) **4.** Solve the differential equation

$$y' + y = xy^{3/2}.$$

(10) **5.** Write the general solution of the following Cauchy-Euler equation

$$x^2y'' + xy' + y = \ln(x)$$

Hint: Use variation of parameters to find the particular solution, and do not forget to put the equation under standard form.

(12) **6.** (a) Find the general solution of the following fourth order differential equation coefficients using the technique of undetermined coefficients

$$y^{(4)} - y = 15e^{2x} + x^2$$

(b) Find the solution that satisfies the initial value problem

$$y(0) = 1\,, \qquad y'(0) = 2\,, \qquad y''(0) = 2\,, \qquad y'''(0) = 8\,.$$

(8) **7. A thermometer is removed from a room where the air temperature is $30°C$, and brought outside, where the temperature is $10°C$. After 1 minute, the thermometer reads $10(2e^{-1} + 1)°C$. What is the reading of temperature after 2 minutes? How long will it take for the thermometer to reach $12°C$?**

(8) **8. Find the radius and the interval of convergence of the power series**

$$\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{n}.$$

(10) **9.** Consider the nonlinear differential equation

$$\frac{1}{1+x}\,y'' - y = 0.$$

(a) Find its singular points, if any.

(b) Find the first 4 non-zero terms of the power series solutions around $x = 0$.

(16) **10.** Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + 8y + 12t \\ \frac{dy}{dt} &= x - y + 12t \end{aligned}$$

(a) Write the system under matrix form.

(b) Solve the homogeneous system associated to the system above.

(c) Use (b) and variation of parameters to solve the system above.

[redacted]

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Concordia University

Final Examination - EMAT 213 - All sections

Date: April 2004
Time Allowed: 3 hours
Instructors: D. Dryanov, A. Keviczky, G. Pusztai
Course Examiner: C. David
Directions: Answer all questions. No calculator and no documentation allowed.

MARKS

(36) **1.** Solve the first-order differential equations

(a) $\sqrt{y^2+1}\,dx = xy\,dy$

(b) $xy' + (1+x)y = e^{-x}\sin 2x$

(c) $\left(1 + \ln x + \dfrac{y}{x}\right)dx + (1 + \ln x)\,dy = 0$

(d) $xy^2\dfrac{dy}{dx} = y^3 - x^3$

(9) **2.** Solve the differential equation

$$yy'' = y' - (y')^2$$

Hint: Use the substitution $u = y'$ which implies that $y'' = \dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx} = u\dfrac{du}{dy}$ to reduce the order.

(9) **3.** Solve the initial-value problem

$$y''' - 6y' = 0, \quad y(0) = y'(0) = y''(0) = 1.$$

(9) **4.** Find the general solution of the differential equation

$$x^2y'' - 4xy' + 6y = x.$$

(10) **5.** Find the general solution of the differential equation

$$y'' - 5y' + 6y = 12x^2 - 2x + e^{2x}.$$

(9) **6.** Find the radius and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^2\,(x-1)^n}{3^n}.$$

(9) **7.** Find the first 4 non-zero terms of the power series solution around $x = 0$ of the differential equation

$$(x^2+2)\,y'' + 3x\,y' - y = 0.$$

(9) **8.** Find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 5x - 4y \\ \frac{dy}{dt} &= x + 2z \\ \frac{dz}{dt} &= 2y + 5z \end{aligned}$$

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Concordia University

EMAT 213 - Final Exam

Instructors: Bertola, Keviczky, Korotkin

Course Examiner: C. David

Date: April 2003.

Time allowed: 3 hours.

Directions: NO CALCULATORS.

[30 pts] Problem 1

Solve the following *first* order ODEs by finding the general solution and the solution of the IVP (when given).

(a) $\left(\frac{2x}{x^2+y^2} + x^2 - y\cos(xy)\right)\mathrm{d}x + \left(\frac{2y}{x^2+y^2} - y - x\cos(xy)\right)\mathrm{d}y = 0$

(b) $\sqrt{x^6+1}\, y' = \frac{6x^5}{y^2}$, $\quad y(0) = 2$

(c) $y' = 2 + e^{y-2x+3}$ [*Hint: substitution of the form* $u = Ax + By + C$]

[20 pts] Problem 2

Solve the following linear ODEs by finding the general solution and the solution of the IVP (when given).

(a) $\frac{\mathrm{d}^4 y}{\mathrm{d}x} - y = 15\cos(2x)$;

(b) $x^2y'' - xy' + y = x^2\ln(x)$, $\quad (x > 0)$, $\quad \begin{cases} y(1) = 0 \\ y'(1) = -2 \end{cases}$

[20 pts] Problem 3

(a) Find the center, the radius of convergence and the interval of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n^2}$$

(b) Find the general solution as a power series centered at the regular point $x = 0$ and up to degree 4 of the differential equation below.

$$y'' + (x-1)y = 0$$

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[10 pts] Problem 4

Consider the following *nonhomogeneous linear* system of ODEs

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y + e^{2t} \\ \dfrac{dy}{dt} = x - y \end{cases}$$

(a) Rewrite the system in matrix form.
(b) Find the general solution, X_c of the associated homogeneous system.
(c) Find a particular solution X_p of the system, using the technique of variation of parameters.
(d) Find the solution of the IVP

$$x(0) = 0\ , \quad y(0) = 0$$

[10 pts] Problem 5

Find the general solution of the following *homogeneous* linear system

$$\begin{cases} x' = -2x + y \\ y' = \quad\quad -y + z \\ z' = \quad\quad -y - z \end{cases}$$

[10 pts] Problem 6

A spring is stretched by one meter by a force of half Newton.
A mass of 4 Kilograms is attached to the loose end and it is released half a meter below the equilibrium position without any initial velocity. Find the motion of the mass knowing that there is a damping coefficient equal to 2 $N \times s/m$.

[10 pts] Bonus Problem

For any diagonalizable matrix **A** and any (sufficiently regular) function $f(x)$ one can define the matrix $f(\mathbf{A})$. Indeed, if **D** is the diagonal form of the matrix **A** with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and **P** is the diagonalizing matrix (so that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$) then

$$f(\mathbf{A}) := \mathbf{P} \begin{pmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & f(\lambda_n) \end{pmatrix} \mathbf{P}^{-1}$$

This is exactly how the exponential can be defined for diagonalizable matrices, using $f(x) = e^x$.
The problem is then to compute the cosine of the matrix (i.e. now using $f(x) = \cos(x)$)

$$\mathbf{A} = \begin{pmatrix} 0 & \pi \\ \frac{\pi}{4} & 0 \end{pmatrix}$$

$$\cos(\mathbf{A}) = \ldots\ldots?$$

8

EMATH C-213. F03/1

Final Solutions. APRIL 2003.

1. A. $\left(\frac{2x}{x^2+y^2} + x^2 - y\cos(xy)\right)dx + \left(\frac{2y}{x^2+y^2} - y - x\cos(xy)\right)dy$

$$\frac{\partial P}{\partial y} = \frac{-4xy}{(x^2+y^2)^2} - \left(y(-\sin(xy))(x) + \cos(xy)\right)$$

$$\frac{\partial Q}{\partial x} = \frac{-4xy}{(x^2+y^2)^2} - \left(x(-\sin(xy))y + \cos(xy)\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \text{Exact.}$$

$$Q = \int\left(\frac{2x}{x^2+y^2} + x^2 - y\cos(xy)\right)dx$$

$$Q = \ln(x^2+y^2) + \frac{x^3}{3} - \frac{y\sin(xy)}{y} + g(y)$$

$$Q_y = \frac{2y}{x^2+y^2} - \cos(xy)x + g'(y) = \frac{2y}{x^2+y^2} - y - x\cos(xy)$$

$\therefore$

$$g'(y) = -y \Rightarrow g(y) = -\frac{y^2}{2}.$$

$\therefore$

$$\ln(x^2+y^2) + \frac{x^3}{3} - \sin(xy) - \frac{y^2}{2} = C.$$

B. $(x^6+1)^{1/2}\, y' = \dfrac{6x^5}{y^2}$; $y(0) = 2$

$$(x^6+1)^{1/2} \frac{dy}{dx} = \frac{6x^5}{y^2}$$

$$y^2\, dy = \frac{6x^5\, dx}{(x^6+1)^{1/2}}$$

$$\int y^2\, dy = \int \frac{6x^5\, dx}{(x^6+1)^{1/2}}$$

$$\frac{y^3}{3} = 2(x^6+1)^{1/2} + c_1$$

$$y = \left(6(x^6+1)^{1/2} + 3c_1\right)^{1/3}.$$

Particular Solution: $y(0) = 2.$

$$2 = (6 + 3c_1)^{1/3}$$

$$8 = 6 + 3c_1$$

$$c_1 = +\frac{2}{3}$$

$$\Rightarrow \quad y = \left(6(x^6+1)^{1/2} + 2\right)^{1/3}.$$

c. $y' = 2 + e^{(y-2x+3)}$

Let $u = y - 2x + 3 \Rightarrow y = u + 2x - 3$

$$\frac{dy}{dx} = u' + 2$$

$\therefore$

$$u' + 2 = 2 + e^{u}$$

$$u' = e^{u}$$

$$\frac{du}{dx} = e^{u}$$

$$e^{-u}\,du = dx$$

$$\int e^{-u}\,du = \int dx$$

$$-e^{-u} = x + c_1$$

$$e^{-u} = -x - c_1$$

$$-u = \ln(-x - c_1)$$

$$u = -\ln(-x - c_1)$$

$$y - 2x + 3 = -\ln(-x - c_1)$$

$$y = 2x - \ln(-x - c_1) - 3.$$

2. A. $y^{IV} - y = 15\cos(2x)$

Using Undetermined coefficients.

$y = y_h + y_p$.

y_h: $y^{IV} - y = 0$

$\lambda^4 - 1 = 0 \Rightarrow \lambda = \pm 1$.

$\therefore y_h = c_1 e^{-x} + c_2 e^{x}$.

y_p:

$y_p = A\cos(2x) + B\sin(2x)$

$y_p' = -2A\sin(2x) + 2B\cos(2x)$

$y_p'' = -4A\cos(2x) - 4B\sin(2x)$

$y_p''' = 8A\sin(2x) - 8B\cos(2x)$

$y_p^{IV} = 16A\cos(2x) + 16B\sin(2x)$

Substituting:

$(16A\cos(2x) + 16B\sin(2x)) - (A\cos(2x) + B\sin(2x)) = 15\cos$

$15A\cos(2x) + 15B\sin(2x) = 15\cos(2x)$

$\Rightarrow A = 1$
$B = 0 \Rightarrow$ Sol.: $y = c_1 e^{-x} + c_2 e^{x} + \cos(2$

F03/5

B. $x^2 y'' - xy' + y = x^2 \ln(x) \quad (x > 0) \quad y(1) = 0;\ y'(1) =$

y_h:

$x^2 y'' - xy' + y = 0$

$m(m-1) - m + 1 = 0$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$; multiplici 2.

$\therefore$

$y_h = c_1 x + c_2 x \ln x.$

y_p:

$y_p = v_1 x + v_2 x \ln x$

Wronskian: $\left[\begin{array}{cc|c} x & x\ln x & 0 \\ 1 & (1+\ln x) & \ln x \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \ln x & 0 \\ 0 & 1 & \ln x \end{array}\right] \rightarrow$

$\left[\begin{array}{cc|c} 1 & 0 & -(\ln x)^2 \\ 0 & 1 & \ln x \end{array}\right] \Rightarrow \begin{array}{l} v_1' = -(\ln x)^2 \\ v_2' = \ln x \end{array} \Rightarrow$

$v_1 = -x(\ln x)^2 + 2x \ln x + 2x$

$v_2 = x \ln x - x$

$\therefore$

$y_p = [-x(\ln x)^2 + 2x \ln x + 2x]x + [x \ln x - x] x \ln x$

$\therefore$

$y = c_1 x + c_2 x \ln x - x^2(\ln x)^2 + 2x^2 \ln x + 2x^2 + x^2(\ln x)^2 - x^2 \ln x$

$y = c_1 x + c_2 x \ln x + x^2 \ln x + 2x^2.$

3.A. $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n^2}$.

$a_n = \frac{3^n (x-2)^n}{n^2}$

R: $\lim_{n \to \infty} \left| \frac{a_{(n+1)}}{a_n} \right| < 1$

$\lim_{n \to \infty} \left| \frac{\left(\frac{3^{(n+1)} (x-2)^{(n+1)}}{(n+1)^2} \right)}{\left(\frac{3^n (x-2)^n}{n^2} \right)} \right| < 1$

$\lim_{n \to \infty} \left| \frac{3(x-2)n^2}{(n+1)^2} \right| < 1$

$3|x-2| \lim_{n \to \infty} \left| \frac{n^2}{(n+1)^2} \right| < 1$

$3|x-2|(1) < 1$

R: $|x-2| < \frac{1}{3}$.

$\therefore$,

Center is: 2.

Radius is: $\frac{1}{3}$.

Interval: $\frac{5}{3} < x < \frac{7}{3}$.

Checking endpoints:

Let $x = \frac{7}{3}$:

$$\sum_{n=1}^{n=\infty} \frac{3^n\left(\frac{7}{3}-2\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges by integral test.

For $x = 5/3$:

$$\sum_{n=1}^{\infty} \frac{3^n\left(\frac{5}{3}-2\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ diverges by the alternating series test.

$\therefore$ Interval: $\frac{5}{3} < x \le \frac{7}{3}$.

3.B. $y'' + (x-1)y = 0$

Let $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{(n-1)}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{(n-2)}$$

Substituting:

$$\left(\sum_{n=2}^{\infty} n(n-1) c_n x^{(n-2)}\right) + (x-1)\left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{(n-2)} + x \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{(n-2)} + \sum_{n=0}^{\infty} c_n x^{(n+1)} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Let $n = n+2$ Let $n = n-1$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{(n+2)} x^n + \sum_{n=1}^{\infty} c_{(n-1)} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\left(2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{(n+2)} x^n\right) + \left(\sum_{n=1}^{\infty} c_{(n-1)} x^n\right) -$$

$$\left(c_0 + \sum_{n=1}^{\infty} c_n x^n\right) = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)\,c_{(n+2)} + c_{(n-1)} - c_n\right]x^n = 0$$

$\therefore$

$$2c_2 - c_0 = 0$$

$$(n+2)(n+1)\,c_{(n+2)} + c_{(n-1)} - c_n = 0 \qquad n \geq 1.$$

$$c_2 = \frac{c_0}{2}$$

$$c_{(n+2)} = \frac{c_n - c_{(n-1)}}{(n+2)(n+1)} \qquad n \geq 1.$$

Let $c_0 = 1;\ c_1 = 0$

$$c_2 = \frac{1}{2}$$

$$n=1:\ c_3 = \frac{c_1 - c_0}{6} = \frac{-1}{6}$$

$$n=2:\ c_4 = \frac{c_2 - c_1}{12} = \frac{\left(\frac{1}{2}\right) - 0}{12} = \frac{1}{24}$$

$$n=3:\ c_5 = \frac{c_3 - c_2}{20} = \frac{\left(-\frac{1}{6}\right) - \left(\frac{1}{2}\right)}{20} = \frac{-4}{120} = \frac{-1}{30}.$$

$\therefore$

$$y_1 = 1 + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \dots$$

Let $c_0 = 0;\ c_1 = 1$

$$c_2 = 0$$

$$n=1:\ c_3 = \frac{c_1 - c_0}{6} = \frac{1}{6}$$

$$n=2: c_4 = \frac{c_2 - c_1}{12} = \frac{(0) - 1}{12} = \frac{-1}{12}$$

$$n=3: c_5 = \frac{c_3 - c_2}{20} = \frac{(1/6) - (0)}{20} = \frac{1}{120}.$$

$\therefore$,

$$y_2 = x + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots\cdots$$

Hence,

$$y = c_1 y_1 + c_2 y_2.$$

4. $\frac{dx}{dt} = 2x - 2y + e^{2t}$

$\frac{dy}{dt} = x - y$

$$X' = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$$

$$X' = AX + F \Rightarrow X = X_h + X_p.$$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} (2-\lambda) & -2 \\ 1 & (-1-\lambda) \end{pmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) + 2 = 0$$

$\lambda^2 - \lambda = 0 \Rightarrow \lambda = 0; +1.$ Theorem; Part A.

Eigenvectors: For $\underline{\lambda = 0}$:

$$(A - \lambda I)K_1 = (A - \lambda I)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$k_1 - k_2 = 0 \Rightarrow k_1 = k_2 \Rightarrow K_1 = k_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{K_1}.$

For $\lambda = 1$:

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$k_1 - 2k_2 = 0 \Rightarrow k_1 = 2k_2 \Rightarrow K_2 = \begin{bmatrix} 2k_2 \\ k_2 \end{bmatrix} = k_2 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{K_2}.$

$\therefore$

$$X_h: \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$$

OR

$$X_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

X_p:

$$X_p = \Phi(t) \int \Phi^{-1}(t) F(t)\, dt$$

$$\Phi(t) = \begin{pmatrix} 1 & 2e^t \\ 1 & e^t \end{pmatrix}; \quad \Phi^{-1}(t) = \begin{pmatrix} e^t & -2e^t \\ -1 & 1 \end{pmatrix}\left(-e^{-t}\right)$$

$\therefore$

$$X_p = \Phi(t) \int \begin{pmatrix} -1 & 2 \\ e^{-t} & -e^{-t} \end{pmatrix}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} dt$$

F03/13 E

$$X_p = \Phi(t)\int\begin{pmatrix}-e^{2t}\\ e^{t}\end{pmatrix}dt$$

$$= \begin{pmatrix}1 & 2e^{t}\\ 1 & e^{t}\end{pmatrix}\begin{pmatrix}-\frac{1}{2}e^{2t}\\ e^{t}\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{2}e^{2t}+2e^{2t}\\ -\frac{1}{2}e^{2t}+e^{2t}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{3}{2}e^{2t}\\ \frac{1}{2}e^{2t}\end{pmatrix}$$

$$X_p = \begin{pmatrix}3\\1\end{pmatrix}\left(\frac{e^{2t}}{2}\right).$$

$$\therefore\quad X = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix}e^{t} + \begin{pmatrix}3\\1\end{pmatrix}\frac{e^{2t}}{2}.$$

Particular Solution; $x(0)=0$; $y(0)=0$.

$$\left.\begin{aligned}x(0)=0 &= c_1 + 2c_2 + \tfrac{3}{2}\\ y(0)=0 &= c_1 + c_2 + \tfrac{1}{2}\end{aligned}\right\}\quad \begin{aligned}c_2 &= -1.\\ c_1 &= \tfrac{1}{2}\end{aligned}$$

$$\therefore\quad X = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix}e^{t} + \begin{pmatrix}3\\1\end{pmatrix}\frac{e^{2t}}{2}.$$

$\dfrac{105}{14}$ (14)

5. Solve. $x' = -2x + y$
$y' = \quad -y + z$
$z' = \quad -y - z$

$$X' = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} X.$$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} (-2-\lambda) & 1 & 0 \\ 0 & (-1-\lambda) & 1 \\ 0 & -1 & (-1-\lambda) \end{pmatrix} = 0$$

$$[(-2-\lambda)(-1-\lambda)^2 + 0 + 0] - [0 - (-2-\lambda) + 0] = 0$$

$$(-2-\lambda)(-1-\lambda)^2 + (-2-\lambda) = 0$$

$$(-2-\lambda)[(-1-\lambda)^2 + 1] = 0$$

$$(-2-\lambda)[2 + 2\lambda + \lambda^2] = 0 \Rightarrow \lambda = -2;\ -1 \pm i.$$

Eigenvectors: For $\lambda = -2$:

$$(A - \lambda I)K_1 = 0$$

$$(A - \lambda I)\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} k_2 &= 0 \\ k_2 + k_3 &= 0 \\ -k_2 + k_3 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} k_2 &= 0 \\ k_3 &= 0 \end{aligned}\right\} \Rightarrow K_1 = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{K_1}.$$

$$\therefore\ X_1 = K_1 e^{-2t}.$$

For $\lambda = -1+i$:

$$\begin{pmatrix} (-1-i) & 1 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (-1-i)k_1 + k_2 \quad &= 0 \\ -ik_2 + k_3 &= 0 \\ -k_2 - ik_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} k_3 &= ik_2 \\ k_2 &= (1+i)k_1 \end{aligned}\right\} \Rightarrow k_3 = (-1+i)k_1$$

$$\Rightarrow K_2 = \begin{bmatrix} k_1 \\ (1+i)k_1 \\ (-1+i)k_1 \end{bmatrix} = k_1 \underbrace{\begin{bmatrix} 1 \\ (1+i) \\ (-1+i) \end{bmatrix}}_{K_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1-i$:

$$\begin{pmatrix} (-1+i) & 1 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}\cos t - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}\sin t.$$

F03/16 (16)

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}\cos t + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}\sin t.$$

$\therefore$ $X = c_1 X_1 + c_2 X_2 + c_3 X_3.$

$$x = c_1 e^{-2t} + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t$$

$$y = c_2 e^{-t}\cos t - c_2 e^{-t}\sin t + c_3 e^{-t}\cos t + c_3 e^{-t}\sin t$$

$$z = -c_2 e^{-t}\cos t - c_2 e^{-t}\sin t + c_3 e^{-t}\cos t - c_3 e^{-t}\sin t.$$

F03/H II

6. $x'' + 2\lambda x' + \omega^2 x = 0$; $x(0) = .5$; $x'(0) = 0$

$$2\lambda = \frac{\beta}{m} = \frac{2}{4} = \frac{1}{2}$$

$$\omega^2 = \frac{k}{m} = \frac{\left(\frac{F}{s}\right)}{(m)} = \frac{\left(\frac{.5}{1}\right)}{(4)} = \frac{1}{8}$$

$\therefore$

$$x'' + \frac{1}{2}x' + \frac{1}{8}x = 0 \qquad x'(0) = 0;\ x(0) = .5$$

$$8x'' + 4x' + x = 0$$

$$8\lambda^2 + 4\lambda + 1 = 0$$

$$\lambda = -\frac{1}{4} \pm \frac{i}{4} \Rightarrow x = c_1 e^{-1/4 t}\cos\left(\frac{t}{4}\right) + c_2 e^{-1/4 t}\sin\left(\frac{t}{4}\right)$$

Particular solution: $x'(0) = 0$; $x(0) = \frac{1}{2}$;

$$x(0) = \frac{1}{2} = c_1 \Rightarrow c_1 = \frac{1}{2}.$$

$$x'(t) = \left[\frac{1}{2} e^{-1/4 t}\left(-\sin\left(\frac{t}{4}\right)\left(\frac{1}{4}\right)\right) + \frac{1}{2}\left(-\frac{1}{4}\right)e^{-t/4}\cos\left(\frac{t}{4}\right)\right]$$

$$+ \left[c_2 e^{-t/4}\left(\cos\left(\frac{t}{4}\right)\left(\frac{1}{4}\right)\right) + c_2\left(-\frac{1}{4}\right)e^{-t/4}\sin\left(\frac{t}{4}\right)\right]$$

$$x'(0) = 0 = \left[-\frac{1}{8} + \frac{c_2}{4}\right] \Rightarrow c_2 = \frac{1}{2}.$$

$$\therefore x = \frac{1}{2}e^{-t/4}\cos\left(\frac{t}{4}\right) + \frac{1}{2}e^{-t/4}\sin\left(\frac{t}{4}\right).$$

F03/18 (18)

~~Ex.~~ 7. $\det\left(\begin{pmatrix} 0 & \pi \\ \frac{\pi}{4} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$; $f(x) = \cos x$

$\det\begin{pmatrix} -\lambda & \pi \\ \frac{\pi}{4} & -\lambda \end{pmatrix} = 0$

$\lambda^2 - \frac{\pi^2}{4} = 0$

$\lambda^2 = \frac{\pi^2}{4} \Rightarrow \lambda = \pm\frac{\pi}{2} \Rightarrow f\left(\frac{-\pi}{2}\right) = 0$

$f\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \cos A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Concordia University

EMAT 213 - Final Exam

Instructor: Dr. Bertola

Course Examiner: Dr. C. David

Date: June 2003.

Time allowed: 3 hours.

Directions: NO CALCULATORS.

This exam has two pages!

[30 pts] Problem 1

Solve the following *first* order ODEs by finding the general solution and the solution of the IVP (when given).

(a) $\Big(-2x\sin(x^2+y^2)+3x^2\Big)dx+\Big(-2y\sin(x^2+y^2)-2ye^{-y^2}\Big)dy=0$

(b) $(x^2+9)\,y'=\dfrac{x}{y^2}\,,\qquad y(0)=2$

(c) $y'=2+\dfrac{e^{y-2x+3}}{y-2x+3}$ [*Hint: substitution of the form* $u=Ax+By+C$]

[20 pts] Problem 2

Solve the following linear ODEs by finding the general solution

(a) $y^{(3)}-3y''+3y'-y=x^3\,;$

(b) $x^2y''+xy'+y=\ln(x)\,,\qquad (x>0).$

[10 pts] Problem 3

Find the center, the radius of convergence and the **interval of convergence** of the following power series

$$\sum_{n=1}^{\infty}\frac{4^n(x+2)^n}{n^2+3}$$

[10 pts] Problem 4

(i)[8pt] Find the general solution as a power series centered at the regular point $x=0$ and up to degree 5 of the differential equation below.

(ii)[1pt] Find the **singular points** of the equation.

(iii)[1pt] What is the **minimum radius of convergence** of the solution written as a power series?

$$(x-2)y''+y=0$$

100 - 600

(3)

[20 pts] Problem 5

Consider the following *nonhomogeneous linear* system of ODEs

$$\begin{cases} \dfrac{dx}{dt} = & -y \quad +2 \\ \dfrac{dy}{dt} = \; x & \quad\quad +4 \end{cases}$$

(a)[8pt] Rewrite the system in matrix form and find the general solution, X_c of the associated homogeneous system.

(b)[8pt] Find a particular solution X_p of the system, using the technique of variation of parameters.

(c)[4pt] Find the solution of the IVP

$$x(\pi) = 0\,, \quad y(\pi) = 0$$

[10 pts] Problem 6

A spring is stretched by one meter by a force of half Newton.

A mass of 4 Kilograms is attached to the loose end and it is released from the equilibrium position without any initial velocity. Find the motion of the mass knowing that there is a damping coefficient equal to $2\ N \times s/m$ and that the mass is subject to an external force of equation

$$F(t) = 65\sin(t)\,.$$

[5 pts] Bonus Problem 1

Consider the system

$$\begin{cases} x' = & y & \\ y' = & & z \\ z' = & ax + by + cz \end{cases}$$

where a, b, c are constants. Write it as an ODE with constant coefficients of order 3 for the variable $x(t)$.

You do not have to solve it, but answer to the question raised below.

What is then the relation between the auxiliary equation of this ODE and the characteristic polynomial of the matrix defining the system? Motivate your answer by computing both the auxiliary equation and the characteristic polynomial.

[5 pts] Bonus Problem 2

Consider the general first order linear equation

$$y' + P(x)y = f(x)\,.$$

Show all the steps that lead to the formula for the particular solution in terms of the complementary solution (i.e. prove the formula of variation of parameters).

(4)

Midterm Exam Emat 213

October 2005

Instructor: ____________

Time allowed: 1h15min sharp.

Material allowed: no calculators.

[10 points] Problem 1.

Consider the two following first-order ODE's:

$$(a) \quad \frac{dy}{dx} - 2y = 2x^2$$
$$(b) \quad \frac{dy}{dx} - 2x = 2y^2 \ .$$

Find the linear one and explain why the other is not linear.
Find the general solution for the linear equation and solve also the IVP, $y(0) = 2$.

[10 points] Problem 2.

(i) Determine which of the following ODE's is exact.

$$(a) \quad y^4 \mathrm{d}x - (x^2 \cos(y))\mathrm{d}y = 0$$
$$(b) \quad (e^{-xy} + \cos(x))\mathrm{d}x + (xe^{-xy} - 1)\mathrm{d}y = 0$$
$$(c) \quad (2xy - ye^{xy})\mathrm{d}x + (x^2 + 4y^3 - xe^{xy})\mathrm{d}y = 0$$

(ii) Solve the exact ODE in point (i) by expressing (as always) the solution in implicit form and also find the solution of the IVP

$$y(0) = 3 \ .$$

[10 points] Problem 3.

(i) Perform the substitution $y = u^{-1}$ in the following Bernoulli equation and reduce it to a linear ODE.

$$\frac{dy}{dx} - y = e^x y^2 \ .$$

(ii) Perform the substitution $y = xu$ in the ODE which is homogeneous among the following two and reduce it to a separable equation for the unknown function u (it is not required that you solve it).

$$(a) \quad (y^2 - 3xy)\mathrm{d}x - x^2\mathrm{d}y = 0$$
$$(b) \quad (2yx - x^3)\mathrm{d}x - y\mathrm{d}y = 0 \ .$$

3

[10 points] Problem 4.

Find the orthogonal trajectories to the family of curves

$$x^4 + 2y^4 = C \ , \qquad (C > 0)$$