2141.F08 A2 Solns

$\mathcal{F}(\mathbb{R})$

1. Let $\mathcal{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ be the vector space (over $\mathbf{R}$) of all real-valued functions defined on $\mathbf{R}$. Let

$$M = \{f \in \mathcal{F}(\mathbf{R}) \mid \forall x \in \mathbf{R}, f(-x) = f(x)\}$$

be the set of *even* functions and

$$N = \{g \in \mathcal{F}(\mathbf{R}) \mid \forall x \in \mathbf{R}, g(-x) = -g(x)\}$$

be the set of *odd* functions.

a) Show that M is a subspace of $\mathcal{F}(\mathbf{R})$.

1) Let 0 denote the zero function of $\mathcal{F}(\mathbb{R})$. Then, $0 \in \mathcal{F}(\mathbb{R})$

Since $0(x) = 0 = 0(-x)$, $\forall x \in \mathbb{R}$,

2) Suppose $f, g \in \mathcal{F}(\mathbb{R})$. Then, $\forall x \in \mathbb{R}$,

$$(f+g)(x) := f(x) + g(x) = f(-x) + g(-x) =: (f+g)(-x).$$

Hence, $f+g \in \mathcal{F}(\mathbb{R})$

3) Suppose $f \in \mathcal{F}(\mathbb{R})$ and $k \in \mathbb{R}$ is a scalar.

Then, $\forall x \in \mathbb{R}$, $(kf)(x) := k \cdot f(x) = k\, f(-x) = (kf)(-x)$

Hence, $kf \in \mathcal{F}(\mathbb{R})$

Thus, M is a s.s. of $\mathcal{F}(\mathbb{R})$

1. (cont.) Now assume that both M and N are subspaces of $\mathcal{F}(\mathbf{R})$.

1b) Prove that $\mathcal{F}(\mathbf{R}) = M \oplus N$ (*direct* sum). (Use any hints provided in Q. 13, P. 24 of the text.)

We first show that $M \cap N = \{0\}^*$, as follows.

Suppose $f \in M \cap N$. Then, $\forall x \in \mathbb{R}$,

$$f(x) = f(-x) = -f(x).$$

i.e., $2f(x) = 0, \quad \forall x \in \mathbb{R}.$

Hence, $f(x) = 0, \quad \forall x \in \mathbb{R}.$

Thus $f = 0$. Hence $M \cap N = \{0\}$

Now we use the hint to show $\mathcal{F}(\mathbb{R}) = M + N$.

Given any $f \in \mathcal{F}(\mathbb{R})$, we know that, $\forall x \in \mathbb{R}$,

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right).$$

Let $f_{ev}(x) := \frac{1}{2}\left(f(x) + f(-x)\right)$, $f_{odd}(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$.

Then $f_{ev}, f_{odd} \in \mathcal{F}(\mathbb{R})$, indeed, $f_{ev}(-x) = \frac{1}{2}\left(f(-x) + f(-(-x))\right)$

$$= \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$= f_{ev}(x), \quad \forall x \in \mathbb{R},$$

and $f_{odd}(-x) = \frac{1}{2}\left(f(-x) - f(-(-x))\right)$

$$= -\frac{1}{2}\left(f(x) - f(-x)\right) = -f_{odd}(-x).$$

Thus, $f_{ev} \in M$ and $f_{odd} \in N$. Hence, $\mathcal{F}(\mathbb{R}) = M + N$.

Thus, by $(*)$ $\mathcal{F}(\mathbb{R}) = M \oplus N$.

2. Let $\mathcal{P}(\mathbf{Q})$ denote the space of all polynomials functions with coefficients in $\mathbf{Q}$.

a) Prove that $\{1, t, t^2\}$ is linearly independent in $\mathcal{P}(\mathbf{Q})$. Suppose $a \cdot 1 + bt + ct^2 = 0$ in $\mathcal{P}(\mathbb{Q})$. Evaluating at $t=0$, $t=1$, & $t=-1$, we obtain

$$\left.\begin{aligned} a1 + b\cdot 0 + c\cdot 0^2 &= 0 \\ a + b + c &= 0 \\ a - b + c &= 0. \end{aligned}\right\}$$

The augmented matrix of this linear system in a, b, c is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

and since the coeff^n matrix has rank 3 = # variables, this system has the unique sol^n $a=b=c=0$.
Hence, $\{1, t, t^2\}$ is l.i. in $\mathcal{P}(\mathbb{Q})$.

b) Prove that $\{\underbrace{1+t+t^2}_{p_1}, \underbrace{t^2+5}_{p_2}, \underbrace{2t+1}_{p_3}, \underbrace{3t^2+5t}_{p_4}\}$ is linearly dependent in $\mathcal{P}(\mathbf{Q})$. (Use (a))

$5+t^2$, $1+2t$, $5t+3t^2$

Suppose $a_1(1+t+t^2) + a_2(5+t^2) + a_3(1+2t) + a_4(5t+3t^2) = 0$ in $\mathcal{P}(\mathbb{Q})$. Then, $(a_1 + 5a_2 + a_3)1 + (a_1 + 2a_3 + 5a_4)t + (a_1 + a_2 + 3a_4)t^2 = 0$ in $\mathcal{P}(\mathbb{Q})$. But, by (a), $\{1, t, t^2\}$ is l.i. in $\mathcal{P}(\mathbb{Q})$, so

$$\left.\begin{aligned} a_1 + 5a_2 + a_3 &= 0 \\ a_1 + + 2a_3 + 5a_4 &= 0 \\ a_1 + a_2 + 3a_4 &= 0. \end{aligned}\right\}$$

Now, this is a homogeneous system of 3 equations in 4 unknowns, and so has (over $\mathbb{Q}$) at least one non-zero solution for (a_1, a_2, a_3, a_4). That is, there are scalars a_1, a_2, a_3 & a_4 in $\mathbb{Q}$, <u>not</u> <u>all</u> zero, s.t. $a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 = 0$. $\{p_1, p_2, p_3, p_4\}$ is l.d.

2. (cont.) Now Let

$$A = \begin{bmatrix} a & b & c & d \\ n & f & g & h \\ j & k & l & m \end{bmatrix} \in \mathbf{M}_{3\,4}(\mathbf{Q})$$

be a 3 by 4 rational matrix such that rank $A = 3$.

2b) Is $\{\overbrace{a+nt+jt^2}^{q_1}, \overbrace{b+ft+kt^2}^{q_2}, \overbrace{c+gt+lt^2}^{q_3}, \overbrace{d+ht+mt^2}^{q_4}\}$ is linearly independent in $\mathcal{P}(\mathbf{Q})$?

No, since $c_1q_1 + c_2q_2 + c_3q_3 + c_4q_4 = 0$* has the same solns as the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} a & b & c & d & 0 \\ n & f & g & h & 0 \\ j & k & l & m & 0 \end{array}\right]$$

But again, this is a homogeneous linear system with fewer eqns than unknowns, and so * always has non-zero solutions for (c_1, c_2, c_3, c_4). (Here, the fact that the rank of A is 3 is not used.)

2c) Does $\{\overbrace{a+nt+jt^2}^{r_1}, \overbrace{b+ft+kt^2}^{r_2}, \overbrace{c+gt+lt^2}^{r_3}, \overbrace{d+ht+mt^2}^{r_4}\}$ span $\mathcal{P}_2(\mathbf{Q})$? Yes: Let $a_0 + a_1t + a_2t^2 \in \mathcal{P}_2(\mathbb{Q})$ be arbitrary, and consider the equation

**... $c_1r_1 + c_2r_2 + c_3r_3 + c_4r_4 = a_0 + a_1t + a_2t^2$. We show that this always has a soln for (c_1, c_2, c_3, c_4), no matter what the values of a_0, a_1 & a_2, as follows. Using (a) again, we find that the augmented matrix of the linear system obtained from ** is

$$\left[\begin{array}{cccc|c} a & b & c & d & a_0 \\ n & f & g & h & a_1 \\ j & k & l & m & a_2 \end{array}\right].$$

However, as A (the coefficient matrix of this system) has rank 3 = # rows, this system is always consistent. Hence, we can always solve (**) for c_1, c_2, c_3 & c_4. $\therefore \{r_1, r_2, r_3, r_4\}$ spans $\mathcal{P}_2(\mathbb{Q})$

3. Let V be a vector space over $\mathbf{F}$. Prove the following:

a) If $\{v_1, \ldots, v_n\} \subset V$ is linearly dependent, and $v \in V$ is any vector, then $\{v, v_1, \ldots, v_n\}$ is also linearly dependent.

Since $\{v_1, \ldots, v_n\}$ is l.d., $\exists\, (c_1, \ldots, c_n) \neq 0$ in F^n s.t. $\sum_{i=1}^{n} c_i v_i = 0$. But then,

$$0v + \sum_{i=1}^{n} c_i v_i = 0$$

as well, and not all of $0, c_1, c_2, \ldots, c_n$ are zero, since not all of $c_1, c_2, \ldots, c_n$ are zero!

b) If $\{w_1, \ldots, w_n\} \subset V$ is linearly independent, then $\{w_2, \ldots, w_n\}$ is also linearly independent.

Suppose $\sum_{i=2}^{n} a_i w_i = 0$, with $a_i \in F$, $2 \le i \le n$.

Then, $0v_1 + \sum_{i=2}^{n} a_i w_i = 0 + 0 = 0.$

That is, if we set $a_1 = 0$, $\sum_{i=1}^{n} a_i w_i = 0$ $\ast$ But $\{w_1, \ldots, w_n\}$ is l.i., so $\ast$ implies that $a_1 = a_2 = \cdots = a_n = 0$. We knew $a_1 = 0$, but now we know $a_2 = a_3 = \cdots = a_n = 0$, so $\{w_2, w_3, \ldots, w_n\}$ is l.i.

c) If $\{u_1, \ldots, u_n\} \subset V$ spans V, and $u_0 \in V$ is any vector, then $\{u_0, u_1, \ldots, u_n\}$ also spans V.

Let $v \in V$. Since $\{u_1, \ldots, u_n\}$ spans V, $\exists\, (c_1, \ldots, c_n) \in F^n$ s.t. $\sum_{i=1}^{n} c_i u_i = v$. But then, $v = 0u_0 + \sum_{i=1}^{n} c_i u_i$ as well, showing that $v = \sum_{i=0}^{n} c_i u_i \in \text{span}\{u_0, u_1, \ldots, u_n\}$. Hence, $V = \text{span}\{u_0, u_1, \ldots, u_n\}$

4. (cont.) Now suppose V is any vector space over a fiedl $\mathbf{F}$, and that $\{u, v, w\}$ is linearly independent in V.

4c) If $\mathbf{F} = \mathbf{F}_3$, is $\{\underbrace{2u - v + w}_{u_3}, \underbrace{-u + 2v + w}_{u_2}, \underbrace{u + v + w}_{u_1}\}$ linearly independent in V? No,

Since $a_1u_1 + a_2u_2 + a_3u_3 = 0 \quad \dots \Diamond$

$\iff a_1(u+v+w) + a_2(-u+2v+w) + a_3(2u-v+w) = 0$

$\iff (a_1 - a_2 + 2a_3)u + (a_1 + 2a_2 + a_3)v + (a_1 + a_2 + a_3)w = 0$

$\iff$ $a_1 - a_2 + 2a_3 = 0$, and
$a_1 + 2a_2 + a_3 = 0$, and
$a_1 + a_2 + a_3 = 0$, since $\{u,v,w\}$ is l.i.

Hence $\Diamond \iff \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. But we saw in 4a) that $\text{rank}_{\mathbf{F}_3}\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = 2$,

so this system (in 3 variables) has non-trivial solutions for (a_1, a_2, a_3). Hence, $\Diamond$ does too, and so $\{u_1, u_2, u_3\}$ is l.d.

4d) If 3 is invertible in $\mathbf{F}$, is $\{\overbrace{2u - v + w}^{u_3}, \overbrace{-u + 2v + w}^{u_2}, \overbrace{u + v + w}^{u_1}\}$ linearly independent in V? Yes; Arguing as in 4c), $a_1u_1 + a_2u_2 + a_3u_3 = 0$

has the same solⁿ for (a_1, a_2, a_3) as $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$.

But if $\frac{1}{3} \in F$, we saw in 4b) that this system has a coeffⁿ. matrix of rank $3 = \#$ variables, and so has only one solⁿ, namely $a_1 = a_2 = a_3 = 0$. Hence, $\{u_1, u_2, u_3\}$ is l.i. (if $\frac{1}{3} \in F$).

4. Let $A = \{\overset{v_3}{(2,-1,1)}, \overset{v_2}{(-1,2,1)}, \overset{v_1}{(1,1,1)}\}$.

a) If $\mathbf{F} = \mathbf{F}_3$, show that A is linearly dependent in $\mathbf{F}_3^3$

$$\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$ which has rank $2 < 3$.

Hence, $\{v_1, v_2, v_3\}$ is l.d.

b) If 3 is invertible in $\mathbf{F}$, show that A is linearly independent in $\mathbf{F}^3$.

$$B = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$ which has rank 3; Hence, the columns of B, viz. v_1, v_2, v_3 are l.i. in F^3 if $\frac{1}{3} \in F$.

Or: $$\det \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 2 & -1 \end{bmatrix} = \begin{vmatrix} 3 & -3 \\ 2 & -1 \end{vmatrix} = 3 \neq 0$$ (in F!)

$\therefore$ the columns are l.i.

5. (Bonus) Prove carefully that $V = \mathbf{R}$ is not finitely generated over $\mathbf{F} = \mathbf{Q}$. (Theorems proven in class may be used.) We know from class that if V is finitely generated over $\mathbb{Q}$, say, by N vectors, then the maximal size of any l.i. set is N. We now show that there are l.i. subsets of $\mathbb{R}$ (over $\mathbb{Q}$) of arbitrarily large size. Indeed, let $P_n = \{\log \underset{p_1}{2}, \log \underset{p_2}{3}, \dots, \log p_n\}$ denote the (natural) logs of the 1st n primes, for any $n \in \mathbb{N}$. We show P_n is l.i. in $\mathbb{R}$ (over $\mathbb{Q}$) as follows: Suppose $\sum_{i=1}^{n} c_i \log p_i = 0$ in $\mathbb{R}$, with $c_i \in \mathbb{Q}$

and p_i denoting the ith prime. Let $c = \operatorname{lcm}(d_1, d_2, \dots, d_n)$, (least common multiple)

(or, simply choose the product) where $d_1, \dots, d_n$ denote the (if the lcm bothers you)

(w.l.o.g) positive denominators of $c_1, \dots, c_n$ respectively.

Then, the equation above implies $\sum_{i=1}^{n} c \cdot c_i \log p_i = 0$. But

now, $c \cdot c_i \in \mathbb{Z}$, for $1 \le i \le n$. So denote $c \cdot c_i =: a_i \in \mathbb{Z}$.

Hence, we have $\sum_{i=1}^{n} a_i \log p_i = 0$. Since $e^{\sum_{i=1}^{n} r_i} = \prod_{i=1}^{n} e^{r_i}$

for real numbers $r_1, \dots, r_n$, we have $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} = 1$.

Write $\{a_1, a_2, \dots, a_n\} = \{a_{i_1}, \dots, a_{i_k}\} \cup \{a_{j_1}, \dots, a_{j_{n-k}}\}$, where

$a_{i_\ell} > 0$, $1 \le \ell \le k$ and $a_{j_m} < 0$ for $1 \le m \le n-k$. Then, we have

$$p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} \cdots p_{i_k}^{a_{i_k}} = p_{j_1}^{-a_{j_1}} \cdots p_{j_{n-k}}^{-a_{j_{n-k}}}$$ is an equation

in $\mathbb{N}$. But the factorization of natural numbers into products of primes is unique (up to a change in order of the factors), so $a_1 = a_2 = \dots = a_n = 0$.