Solution to Midterm Examination 2 (version A)

MAT 1322-3X, Summer 2014

1. (5 marks) Consider the initial-value problem $\frac{dy}{dt} = \frac{\cos t}{e^{2y}}$, $y(0) = 0$.

(a) (2 marks) Use Euler's method to find and approximation of $y(0.3)$ with step-size $h = 0.1$. (Use at least 6 digits after the decimal point in your calculation.)
(b) (3 marks) Solve this initial-value problem to find the exact value of $y(0.3)$.

Solution. (a) $t_0 = 0$, $y(0) = y_0 = 0$.

$t_1 = 0.1$, $y(0.1) \approx y_1 = y_0 + h\frac{\cos t_0}{e^{2y_0}} = 0 + 0.1 \times \frac{\cos 0}{e^0} = 0.1$.

$t_2 = 0.2$, $y(0.2) \approx y_2 = y_1 + h\frac{\cos t_1}{e^{2y_1}} = 0.1 + 0.1 \times \frac{\cos 0.1}{e^{0.2}} \approx 0.181464$.

$t_3 = 0.3$, $y(0.3) \approx y_3 = y_2 + h\frac{\cos t_2}{e^{2y_2}} = 0.181464 + 0.1 \times \frac{\cos 0.2}{e^{0.362928}} \approx 0.249641$.

(b) $e^{2y}\, dy = \cos t\, dt$. $\frac{1}{2}e^{2y} = \sin t + C$. $\frac{1}{2}e^0 = \sin 0 + C$, $C = \frac{1}{2}$. Then $\frac{1}{2}e^{2y} = \sin t + \frac{1}{2}$,
$e^{2y} = 2 \sin t + 1$. $y = \frac{1}{2}\ln(2 \sin t + 1)$. $y(0.3) = \frac{1}{2}\ln(2 \sin 0.3 + 1) \approx 0.232194$.

2. (4 marks) Suppose salted water of concentration 5 g / m^3 is added to a reservoir of volume 100 m^3 at a rate 2 m^3/ minute. Assume the water in the reservoir is well mixed and the same amount of mixed water is removed from the reservoir. The reservoir is originally filled with fresh water (i.e., concentration of salt is 0 g / m^3). Let $Q(t)$ be the quantity, in grams, of salt in the reservoir at time t.

(a) (2 marks) Find the differential equation that $Q(t)$ satisfies. What is the initial condition?
(b) (2 marks) Solve this equation analytically.

Solution. (a) Rate of increasing $r_{\text{inc}} = 5 \times 2 = 10$ g / min. Rate of decreasing $r_{\text{dec}} = (Q / 100) \times 2 = 0.02Q$ g / min. The differential equation is $\frac{dQ}{dt} = 10 - 0.02Q$. The initial condition is $Q(0) = 0$.

(b) $\frac{dQ}{dt} = 10 - 0.02Q$, $Q(0) = 0$. $\int \frac{1}{10 - 0.02Q}dQ = dt, -\frac{1}{0.02}\ln|10 - 0.02Q| = t + C$.

$|10 - 0.02Q| = K_1e^{-002t}$, where $K_1 = e^{-0.02C} > 0$. $10 - 0.02Q = K_2e^{-0.02t}$, where $K_2 = \pm K_1 \neq 0$. By the initial condition, $K_2 = 10$. Hence, $10 - 0.02Q = 10e^{-0.02t}$. $Q = 500(1 - e^{-0.02t})$.

3. (4 marks) Suppose a piece of metal is brought into a room with temperature 25°C at 2 pm. At 4 pm, its temperature is 115°C, and at 5 pm, its temperature is 55°C. What is its temperature when it is brought into the room?

Solution. $T(t) = T_e + (T_0 - T_e)e^{-kt}$, where k is the time after 2 pm. $T(1) = 25 + (T_0 - 25)e^{-2k} = 115$, $T(3) = 25 + (T_0 - 25)e^{-3k} = 55$. $(T_0 - 25)e^{-2k} = 90$, $(T_0 - 25)e^{-3k} = 30$. $\dfrac{(T_0-25)e^{-2k}}{(T_0-25)e^{-3k}} = \dfrac{90}{30}$, $e^k = 3$, $e^{2k} = 9$. The temperature at 2 pm is $T_0 = 25 + 90e^{2k} = 25 + 90 \times 9 = 835$°C.

Alternative solution: $T(t) = T_e + (T_0 - T_e)e^{-kt}$, where k is the time after 4 pm. Then $T_0 = 115$. $T(1) = 25 + (T_0 - 25)e^{-k} = 25 + 90e^{-k} = 55$. $e^k = 3$.

The temperature at 2 pm is $T(-2) = 25 + (115 - 25)e^{2k} = 25 + 90 \times 9 = 835$°C.

4. (4 marks) Consider the initial-value problem:

$$dy / dt = 5y - y^2.$$

(a) (1 mark) Find the equilibrium solutions of this equation.
(b) (3 marks) Solve this problem analytically with initial condition $y(0) = 2$. (You must show how the solution is obtained step by step).

Solution. (a) Let $5y - y^2 = 0$. We have two equilibrium solutions $y = 0$, $y = 5$.

(b) Since $\dfrac{1}{y(5-y)} = \dfrac{1}{5}\left(\dfrac{1}{y} + \dfrac{1}{5-y}\right)$, $\displaystyle\int \frac{1}{y(5-y)}dy = \frac{1}{5}\ln\left|\frac{y}{5-y}\right| = t + C$.

$\left|\dfrac{y}{5-y}\right| = K_1 e^{5t}$, where $K_1 = e^{5C} > 0$. Removing the absolute value sign, $\dfrac{y}{5-y} = K_2 e^{5t}$, where $K_2 = \pm K_1 \neq 0$. When $t = 0$, $y = 2$, $K_2 = \dfrac{2}{3}$. $\dfrac{y}{5-y} = \dfrac{2}{3}e^{5t}$. $y = (5 - y)\left(\dfrac{2}{3}e^{5t}\right)$.

$3y = 10e^{5t} - 2ye^{5t}$. $(3 + 2e^{5t})y = 10e^{5t}$. $y = \dfrac{10e^{5t}}{3+2e^{5t}} = \dfrac{10}{3e^{-5t}+2}$.

5. (3 marks) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{5^n}$.

Solution. The first term $a_0 = 1$. The common ratio is $r = \dfrac{a_{n+1}}{a_n} = \dfrac{(-1)^{n+1}2^{2n+2}}{5^{n+1}} \dfrac{5^n}{(-1)^n 2^{2n}} = -\dfrac{4}{5}$.

The sum is $S = \dfrac{1}{1+\frac{4}{5}} = \dfrac{5}{9}$.