Solution to Midterm Examination 1 (version A)

MAT 1322-3X, Summer 2014

1. (3 marks) Use the definition of improper integrals to find the value of improper integral $\int_2^3 \frac{1}{\sqrt{3x-6}}\,dx$.

Solution. (i) $\int_2^3 \frac{1}{\sqrt{3x-6}}\,dx = \lim_{a\to 2^+}\int_a^3 \frac{1}{\sqrt{3x-6}}\,dx$.

Use variable substitution, $u = 3x - 6$, $u' = 3$.

$$\int_2^3 \frac{1}{\sqrt{3x-6}}\,dx = \lim_{a\to 2^+}\int_a^3 \frac{1}{\sqrt{3x-6}}\,dx = \lim_{a\to 2^+}\frac{1}{3}\int_{3a-6}^3 \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\lim_{a\to 2^+}\left[2\sqrt{u}\,\right]_{u=3a-6}^3 = \frac{2\sqrt{3}}{3}$$

2. (3 marks) Use the comparison test to show that improper integral $\int_1^\infty \frac{x}{\sqrt{2x^3-1}}\,dx$ diverges.

Solution. Since $2x^3 - 1 < 2x^3$. $\frac{x}{\sqrt{2x^3-1}} > \frac{x}{\sqrt{2x^3}} = \frac{1}{\sqrt{2x}}$. Since $\int_1^\infty \frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int_1^\infty x^{-1/2}dx$ diverges, improper integral $\int_1^\infty \frac{x}{\sqrt{2x^3-1}}\,dx$ diverges.

3. (3 marks) Find the area of the region bounded by the graph of $y = x^2 - 2x + 1$ and the graph of $y = -2x^2 + 4x + 1$.

Solution. Let $x^2 - 2x + 1 = -2x^2 + 4x + 1$. $3x^2 - 6x = 0$. $x = 0, 2$. The area of this region is

$$A = \int_0^2 ((-2x^2+4x+1)-(x^2-2x+1))dx = \int_0^2 (-3x^2+6x)dx = \left[-x^3+3x^2\right]_{x=0}^2 = 4.$$

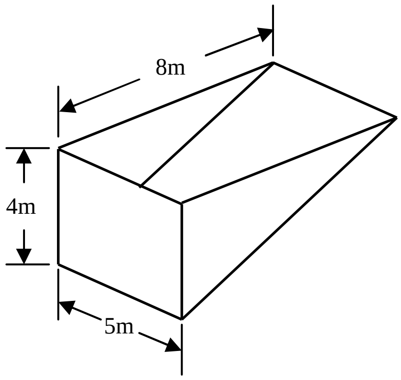
4. (3 marks) Let R be the region under the graph of $y = \arcsin x$ and above the x-axis, $0 \le x \le 1$. Find the volume of the solid obtained by revolving R about the ***y*-axis**.

Solution. The inverse of this function is $x = \sin y$, $0 \le y \le \frac{\pi}{2}$. The volume of the solid is

$$V = \pi\int_0^{\pi/2} (1^2 - \sin^2 y)dy = \pi\int_0^{\pi/2} \cos^2 y\,dy = \frac{\pi}{2}\int_0^{\pi/2}(1+\cos(2y))dy$$

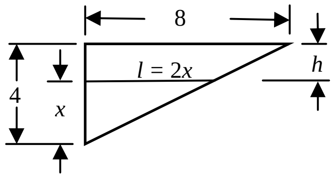
$$= \frac{\pi}{4}\int_0^{\pi}(1+\cos u)du = \frac{\pi}{4}\left[u+\sin u\right]_{u=0}^{\pi} = \frac{\pi^2}{4}.$$

5. (4 marks) A tank is of the shape shown in the following figure:



The tank is filled with water with density $\rho = 1000$ kg/m^3. Find the work needed to pump all the water to a level 2 meters above the top of the tank. (Use $g = 9.8$ m/sec^2).

Solution. Look at a layer of water in the tank x meters above the bottom of the tank. The width of this rectangular layer is found by $\frac{l}{x} = \frac{8}{4}$, $l = 2x$. This area of the layer is $5 \times (2x) = 10x$ m^2.



Let the thickness of the layer be dx. Then the volume of the layer is $dV = 10xdx$, and the weight of the layer is $dw = 10x\rho g dx$. The work needed to pump this layer of water to 2 meters above the tank is $dW = 10x\rho g(6 - x)\,dx$.

Finally, the total work needed to empty the tank is

$$W = \int_0^4 10x\rho g(6-x)dx = 10\rho g\left[3x^2 - \frac{x^3}{3}\right]_{x=0}^4 = \frac{800}{3}\rho g \approx 2.6\times 10^6 \text{ Joule.}$$

If you choose to use the distance between the layer and top of the tank as the variable, denoted by, say, h, then the length of the rectangular layer is found by $\frac{l}{4-h} = \frac{8}{4}$, $l = 2(4 - h)$. This area of the layer is $5 \times 2(4 - h) = 10(4 - h)$ m^2.

Let the thickness of the layer be dh. Then the volume of the layer is $dV = 10(4 - h)dh$, and the weight of the layer is $dw = 10(4 - h)\rho g dh$. The work needed to pump this layer of water to 2 meters above the tank is $dW = 10(4 - h)\rho g(2 + h)dh$.

Finally, the total work needed to empty the tank is

$$W = \int_0^4 10(4-h)\rho g(2+h)dh = \int_0^4 10\rho g(8+2h-h^2)dh = 10\rho g\left[8h+h^2-\frac{h^3}{3}\right]_{h=0}^{4} = \frac{800}{3}\rho g$$

$\approx 2.6\times 10^6$ Joule.

6. (4 marks) Let R be the region between the graph of $y = \frac{1}{x}$ and the x-axis, $1 \le x \le 2$. Assuming it has a uniform density ρ. Find the coordinates of the center of mass of this region.

Solution. The mass of this region is $\rho\int_1^2 \frac{1}{x}dx = \rho \ln 2$. The moment $M_y = \rho \int_1^2 x\left(\frac{1}{x}\right)dx = \rho$.

The moment $M_x = \frac{\rho}{2}\int_1^2\left(\frac{1}{x}\right)^2 dx = \frac{\rho}{4}$. Hence, the coordinates of the center of mass is

$\overline{x} = \frac{M_y}{m} = \frac{1}{\ln 2}$, and $\overline{y} = \frac{M_x}{m} = \frac{1}{4\ln 2}$.