2141. F05 A1 solns

1. Write $\frac{1}{(x^3+x)}$ in the form

$$\frac{1}{(x^3+x)} = \frac{a}{x} + \frac{b}{x+i} + \frac{c}{x-i},$$

with $a, b, c \in \mathbf{C}$.

$$\frac{a}{x} + \frac{b}{x+i} + \frac{c}{x-i} = \frac{a}{x} + \frac{b(x-i)+c(x+i)}{(x+i)(x-i)}$$

$$= \frac{a}{x} + \frac{(b+c)x + (c-b)i}{x^2+1}$$

$$= \frac{a(x^2+1) + x[(b+c)x + (c-b)i]}{x(x^2+1)}$$

$$= \frac{(a+b+c)x^2 + (c-b)ix + a}{x^3+x}.$$

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Hence, we need

$$\begin{aligned} a+b+c &= 1 \\ b-c &= 0 \\ a &= 1. \end{aligned}$$

The unique solution is $(a, b, c) = (1, -\frac{1}{2}, -\frac{1}{2})$

Thus $$\frac{1}{x^3+x} = \frac{1}{x} - \frac{1}{2(x+i)} - \frac{1}{2(x-i)}$$

2. a) Find the conditions that $a, b, c \in \mathbf{Q}$ must satisfy so that the system

$$\begin{array}{rrrrcl} x_1 & -2x_2 & +x_3 & +2x_4 & = & a \\ x_1 & +x_2 & -x_3 & +x_4 & = & b \\ x_1 & +7x_2 & -5x_3 & -x_4 & = & c \end{array}$$

has at least one solution in $\mathbf{Q}^4$.

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 1 & 1 & -1 & 1 & b \\ 1 & 7 & -5 & -1 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 0 & 3 & -2 & -1 & b-a \\ 0 & 9 & -6 & -3 & c-a \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 0 & 3 & -2 & -1 & b-a \\ 0 & 3 & -2 & -1 & \frac{c-a}{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 0 & 3 & -2 & -1 & b-a \\ 0 & 0 & 0 & 0 & \frac{c-a}{3} - (b-a) \end{array}\right]. \quad 2 \quad *$$

But $\frac{c-a}{3} - b + a = \frac{2a}{3} - b + \frac{c}{3}$.

The system above is consistent $\Leftrightarrow$ $\frac{2a}{3} - b + \frac{c}{3} = 0$

i.e. $2a - 3 + c = 0$. 2

2b). Find the conditions that $a, b, c \in \mathbf{F}_3$ must satisfy so that the system

$$\begin{array}{rrrrcl} x_1 & -2x_2 & +x_3 & +2x_4 & = & a \\ x_1 & +x_2 & -x_3 & +x_4 & = & b \\ x_1 & +x_2 & -2x_3 & -x_4 & = & c \end{array}$$

$F_3 = \{0, 1, 2\}$

has at least one solution in $\mathbf{F}_3^4$.

The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 1 & 1 & -1 & 1 & b \\ 1 & 1 & -2 & -1 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & a \\ 1 & 1 & 2 & 1 & b \\ 1 & 1 & 1 & 2 & c \end{array}\right] \quad \left(\begin{array}{l} -2 = 1 \\ -1 = 2 \end{array}\right)$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & a \\ 0 & 0 & 1 & 2 & b-a \\ 0 & 0 & 0 & 0 & c-a \end{array}\right].$$ Hence, this system

is consistent iff $a - c = 0$.

3. Let $A = \begin{bmatrix} \bar{1} & \bar{1} & \bar{1} \\ \bar{1} & \bar{0} & \bar{1} \end{bmatrix} \in \mathbf{M}_{2\,3}(\mathbf{F}_2)$

a) Find the reduced row echelon form of the matrix (working in $\mathbf{F}_2$, of course.)

b) Find all solutions $x \in \mathbf{F}_2^3$ of $Ax = 0$

a) (We'll drop the bars) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. 2

b) From a) $Ax = 0$ has augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$$\sim \begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{l} x = s \\ y = 0 \\ z = s \end{array}, \quad s \in F_2. \quad 1$$

Hence $\{(0,0,0), (1,0,1)\}$ are the only sols in F_2^3.

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4. a) Let $\mathbf{F}$ be any field, and fix $a \in \mathbf{F}$. Equip the set $V = \mathbf{F}^2$ with two operations as follows. Define addition by

$$(x,y)\tilde{+}(x',y') := (x+x', y+y'-a), \quad \forall x,x',y,y' \in F,$$

and define the scalar multiplication by scalars by

$$c \circledast (x,y) := (cx, cy - ac + a), \quad \forall x,y,c \in F.$$

(i) Prove that $\mathbf{F}^2$, with these two operations satisfies the two existence axioms for a vector space over $\mathbf{F}$.

(ii) Prove that $\mathbf{F}^2$, with these two operations satisfies the first distributive property for a vector space over $\mathbf{F}$, i.e. prove that $\forall c \in \mathbf{F}$ and $\forall u,v \in \mathbf{F}^2$,

$$c \circledast (u\tilde{+}v) = (c \circledast u)\tilde{+}(c \circledast v).$$

(i) a) We seek $(x_0,y_0) \in F^2$ s.t. $(x_0,y_0)\tilde{+}(x,y) = (x,y)$

i.e. $(x_0+x, y_0+y-a) = (x,y)$.

Hence, Choose $(x_0,y_0) = (0,a)$. i.e. $(0,a)$ is the zero,

since $(0,a)\tilde{+}(x,y) = (x,y)$, $\forall (x,y) \in F^2$ 2

b) If $(x',y')\tilde{+}(x,y) = (0,a)$, then

$(x'+x, y'+y-a) = (0,a)$, so $x' = -x$, $y' = 2a-y$. 2

Thus, $(-x, 2a-y)\tilde{+}(x,y) = (0,a)$, $\forall (x,y) \in F^2$,

so $-(x,y) = (-x, 2a-y)$ is the negative of (x,y).

(ii) Let $u = (x,y)$ and $v = (x',y')$. Then $c \circledast (u\tilde{+}v) = c \circledast (x+x', y+y'-a)$
$= (cx+cx', cy+cy'-ca-ac+a) = (cx+cx', cy+cy'-2ca+a)$. On the other hand, $(c\circledast u)\tilde{+}(c\circledast v) = (cx, cy-ac+a)\tilde{+}(cx', cy'-ac+a)$
$= (cx+cx', cy-ac+a+cy'-ac+a-a) = (cx+cx', cy+cy'-2ac+a)$,
which is equal to $c\circledast(u\tilde{+}v)$, $\forall c \in F$ and $\forall (x,y),(x',y') \in F^2$. 2

4 b). Let V be the set $\mathcal{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ of all real-valued functions on $\mathbf{R}$. Equip V with the usual addition of functions (for $f, g \in V$, $(f+g)(x) := f(x) + g(x), \forall x \in X$) but with the following operation for multiplication by scalars:

$$(c \ominus f)(x) := f(cx), \quad \forall c, x \in \mathbf{R}.$$

Prove that V with these two operations is *not* a vector space over $\mathbf{R}$. (Show that some axiom fails or that some theorem, true for any vector space, fails.)

If V were a v.s. over $\mathbb{R}$, then $0 \ominus f = 0$, $\forall f \in V$.
(The first zero is the scalar $0 \in \mathbb{R}$, the second is the zero function.) However, let $f(x) = 1$, $\forall x \in \mathbb{R}$
(the constant function 1.) Then,

(1 good choice)

$$(0 \ominus f)(x) = f(0) = 1, \quad \forall x \in \mathbb{R}$$

i.e. $0 \ominus f = f$! But f is not the zero function. (Since e.g. $f(1) = 1 \neq \underset{\text{zero scalar}}{0} = \underset{\text{zero fn.}}{0}(1)$) (1 well-argued)

Hence, V cannot be a v.s. over $\mathbb{R}$.

(OR: $(-1) \ominus f \neq -f$, $\forall f \in V$, since e.g. $g(x) = x^2$ satisfies
$$(-1) \ominus g = g \neq -g \quad \left(\text{e.g. } \begin{matrix} g(1) = 1 \\ -g(1) = -1 \end{matrix} \right).$$

OR ... find an instance of an axiom that fails ...)

5. (Bonus) Let $\mathbf{F}$ be a field. A sequence $v = (a_1, a_2, \dots)$ of elements in $\mathbf{F}$ is said to be *eventually-zero* if $\exists m \in \mathbf{N}$ (which may depend on v) such that $n \geq m \Rightarrow a_n = 0$.

Let

$$V = \{(a_1, a_2, \dots) \mid a_i \in \mathbf{F}\},$$

be the vector space of all sequences in $\mathbf{F}$, and

$$V_1 = \{(a_1, a_2, \dots) \in V \mid (a_1, a_2, \dots) \text{ is eventually-zero}\},$$

and equip it with the same operations as V. The set V, with these operations, is also a vector space over $\mathbf{F}$ (Indeed, V_1 is a subspace of V.)

Define sequences $E_i, i \in \mathbf{N}$, which belong to both V and V_1, by

$$(E_i)_j := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad i \in \mathbf{N}.$$

Now consider

$$\mathcal{F}(V) := \{f \mid f : V \to \mathbf{F}\},$$

and

$$\mathcal{F}(V_1) := \{f \mid f : V \to \mathbf{F}\},$$

both equipped with the usual operations on function spaces, and define elements $f_1, f_2, \dots,$ which belong to both $\mathcal{F}(V)$ and $\mathcal{F}(V)$, by

$$f_i(a_1, a_2, \dots) = a_i, \quad i \in \mathbf{N}.$$

Note that there is a surjective (in fact *bijective*) map $\mathbf{N} \to \{f_i \mid i \in \mathbf{N}\}$ which sends i to f_i.

a) Show that $\{E_i \mid i \in \mathbf{N}\}$ spans V_1.

b) Show that $\{E_i \mid i \in \mathbf{N}\}$ does not span V.

c) Show that $\{f_i \mid i \in \mathbf{N}\}$ does not span $\mathcal{F}(V_1)$.

d) Show that there is no set $W \subset \mathcal{F}(V)$ such that
 (i) W spans $\mathcal{F}(V)$, and
 (ii) There is a surjective map $\mathbf{N} \to W$.

a) $(a_1, a_2, \dots, a_m, 0, 0, \dots) = \sum_{i=1}^{m} a_i E_i$. 1

b) $(1, 1, \dots) \notin \text{span}\{E_i \mid i \in \mathbf{N}\}$ since if $(1, 1, \dots) = \sum_{i=1}^{m} c_i E_i$, then 1 the $n+1^{st}$ element in the sequence on the RHS is 0, while it is not on the LHS.

c) Define $f : V_1 \to F$ by $f(a_1, a_2, \dots) = \sum_{i \in \mathbb{N}} a_i$. Since $(a_1, a_2, \dots) \in V_1$, this sum is actually finite, and so $f \in \mathcal{F}(V_1)$. However, suppose $f \in \text{span}\{f_i \mid i \in \mathbb{N}\}$, say $f = \sum_{i=1}^{n} c_i f_i$. Then $f(E_{n+1}) = \sum_{i=1}^{n} c_i f_i(E_{n+1}) = 0$, since $f_i(E_{n+1}) \forall$ $1 \leq i \leq n$. But $f(E_{n+1}) = 1$. Hence $f \notin \text{span}\{f_i \mid i \in \mathbb{N}\}$. 2

d) See me.