Carleton UNIVERSITY

Final
EXAMINATION
WINTER 2013

DURATION: 3 HOURS **No. Of Students: about 305 expected**

Department Name & Course Number: ECOR 2606 D & E

Course Instructor (s): John Bryant, Bruce Wallace

AUTHORIZED MEMORANDA
Approved calculator and instructor supplied crib sheet

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to a proctor. This question paper has ____10____pages + cover page = _11__ pages in all. The crib sheet has 6 pages (printed double sided).

This examination question paper may not be taken from the examination room.

In addition to this question paper, students require: an examination booklet no
Scantron Sheet no

Name: ____

Student Number: ______________________________________

There are 6 questions worth a total of 100 marks. Attempt all of them.

Answer all questions on the question paper. An extra page has been provided at the end. If you need extra space for any question make a note ("See extra page") and complete your answer on the extra page.

Do NOT write in the boxes below.

Q1	Q2	Q3	Q4	Q5	Q6	Total

Question 1 (16 marks)

i) Demonstrate your understanding of Gaussian elimination by filling in the blank boxes below

original C =

12	12	30	-42
2	3	-4	3
4	-4	3	-23

C after using first pivot value =

12	12	30	-42
0	1	-9	10
0	-8	-7	-9

ii) Using partial pivoting, which value would be used as the next pivot? ____-8______________

iii) Assume that for some system of equations $Aj = b$ forward Gaussian elimination produces the following augmented matrix *C*.

1	-2	6	-10
0	4	2	12
0	0	3	6

What is the solution to the system of equations?

j_1 = ___-18_________________ j_2 = ______2_______________ j_3 = _______2_____________

iv) If the Gauss-Jordan process were applied to the system of equations of part (iii), what would the final augmented matrix *C* be?

1	0	0	-18
0	1	0	2
0	0	1	2

v) Assume that for some system of equations $Aj = b$; A and b are as follows:

A

2	1	-1
-1.5	3	0.75
0.25	0.5	1

b

4
9
-3

If the Gauss-Siedel method were used to solve the system of equations, what would C and d be?

C

0	-1/2	1/2
1/2	0	-1/4
-0.25	-0.5	0

d

2
3
-3

vi) Give the Matlab code required to solve $Ax=b$ using LU factorization (with permutation matrix). You can assume that A and b have been defined and no output is required. What is the benefit of this approach?

```
A=[…];
b=[…]';
[L,U,P]=lu(A);
d=L\(P*b);
x=U\d
fprintf('The solution is %f,%f and %f \n',x) ; % assuming 3 unknowns
```

L,U,P can be reused with other b vectors – eliminating need to repeat Gaussian elimination

Question 2 (16 marks)

Assume a set of data points is stored in vectors x and y.

i) Give all of the Matlab code required to fit a 4th order polynomial to the data and to use this polynomial to estimate y at $x = 4.5$.

```
p = polyfit (x, y, 4)
```

fprintf ('The estimated value of y at x = 4.5 is %7.6f\n', ______polyval(p,4)______________);

ii) Give all of the Matlab code required to fit a curve of the form $y = H x/(K + x)$ to this data and output the values of H and K.

```
y=[….];
x=[…];
invy=1./y;
invx=1./x;
p=polyfit(invx,invy,1);
H=1/p(2);
K=p(1)*H;


fprintf ('H is %f and K is %f\n', __________H,K________________);
```

iii) Suppose that you were asked to fit a curve of the form below (where A, B, C, and D are unknowns to be determined) to the data. Explain how you would go about doing this.

$$ln(y) = (A + Bcos(x) + Cx^2 + Dln(4x))^2$$

Take sqrt of both sides
Change of variable v = sqrt(ln(y))
Do GLS on v = A + B cos(x) + Cx^2 + D ln(4x) to find A,B,C,D

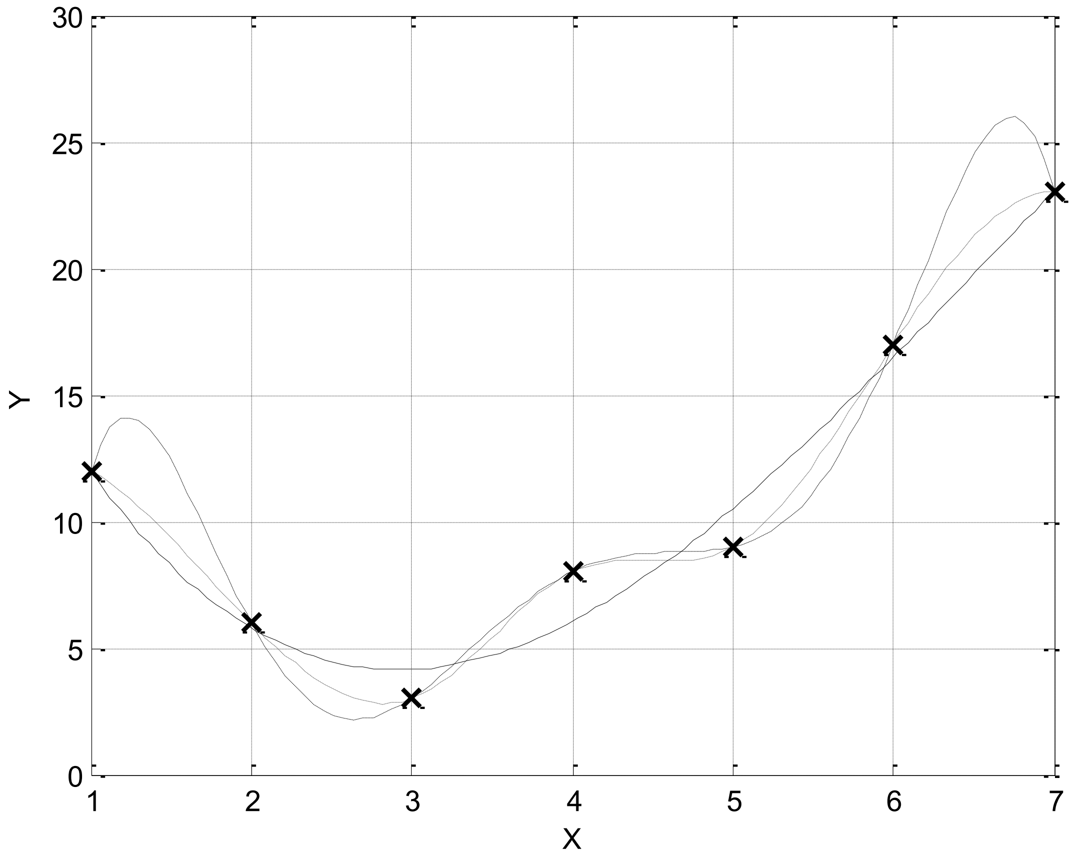
iv) Write a function (to be called *GLS3*) that, given two vectors containing x and y values and three functions (f_1, f_2, and f_3), fits a curve of the form $y = A f_1(x) + B f_2(x) + C f_3(x)$ to the data and returns a vector containing the values of A, B, and C. The code below illustrates how this function might be used.

```
f1 = @(x)  cos(2*x);
f2 = @(x)  sin(2*x);
f3 = @(x) 1;
fit = GLS3 (x, y, f1, f2, f3);
fprintf ('The best fit is %f cos(2x) + %f sin(2x) + %f\n', fit);
```

```
function [ ABC ] = GLS (x, y, f1, f2, f3)
% comments
n=length(x)
if n~=length(y)
error('unequal x and y length')
end
Z=zeros(n,3);
for k=1:n
Z(k,1)=f1(x(1)); Z(k,2)=f2(x(1)); Z(k,3)=f3(x(1));
end
ABC=Z\y;
end
```

Question 3 (18 marks)

The graph below shows a set of data points ('x' markers) and three curves (one solid, one dashed, and one dotted).   .



i) One of the curves is the best fit cubic, one is an interpolating polynomial, and one is a cubic spline. Which is which? There are no marks for guessing – you must clearly justify your answers.

Solid = best fit cubic (does not pass through points)
Dashed = interpolating polynomial (oscillates)
Dotted = spline (all that is left)

ii) Add a linear spline to the graph. No calculations or Matlab code are required - just draw the spline on the graph.

iii) The data points are defined below. Give the additional code required to exactly replicate the graph (without the linear spline but including the labels). You need not worry about the colours used.

```
n = 7; x = 1:n;
y = [ 12 6 3 8 9 17 23];
```

```
pp = spline (x,y);
p = polyfit (x, y, 3);
pn1 = polyfit (x, y, n - 1);
xp = linspace(1, n, 100);
figure (1);
plot (x, y, 'x', xp, polyval (p, xp), xp, ppval (pp, xp), ':', ...
xp, polyval (pn1, xp), '--', 'MarkerSize', 10);
xlabel ('X');
ylabel ('Y');
grid on;
```

iv) Where is the minimum of the cubic spline located? Write code that outputs the location on the minimum and the value of the spline at this point. You may make use of variables from part (iii).

```
f = @(x) ppval (pp, x);
minx = fminbnd (f, 2, 3);
fprintf ('The minimum is located at %f\n', minx);
fprintf ('At this point the value of the spline is %f\n', ppval(pp,minx));
```

v) The best fit cubic and the interpolating polynomial intersect (cross each other) between $x = 5.5$ and $x = 6.5$. Write code that finds and outputs the precise location of one of this intersection. You may make use of variables from part (iii).

```
f = @(x) polyval(p, x) - polyval(pn1,x);
intX = fzero (f, [5.5, 6.5]);
fprintf ('The curves cross at x = %f\n, intX);
```

Question 4a (8 marks)

Values for some function $y=f(x)$ are given below:

x	0.0	0.5	1.0	1.5	2.0
y	2.0	10.1	15.7	17.1	11.7

Estimate the $Z = \int_0^2 f(x)\,dx$ using all of the data points provided

i) trapezoidal integration

(0.5/2)*(2+11.7+2*(10.1+15.7+17.1)) = 24.8750

ii) Simpson's 1/3 rule

(0.5/3)*(2+11.7+2*15.7+4*(10.1+17.1)) = 25.6500

Question 4b (8 marks)

A fluid flows through a pipe. The flow varies with time. We want to estimate the volume of fluid (V, in litres) that passes through the pipe between time t = 10 secs and t = 14 seconds (i.e. we want to integrate the flow between these times).

The available instrumentation allows us to take measure the instantaneous flow rate (in L/s) at any three times of our choosing. We might, for example, decide to measure the flow at t = 10 secs, t = 12 secs, and t = 14 secs. At what three times would you choose to measure the flow? Keep in mind that we want to produce the best possible estimate (second best solutions will receive minimal marks).

t_1 = ____10.4508sec ________ t_2 = ________12 sec________ t_3 = ____13.5492sec ________

Assume that the results are the measurements are $M(t_1)$, $M(t_2)$, and $M(t_3)$ L/s. Give an expression for the volume of fluid that passes through the pipe in the period of interest.

V = ____________((14-10)/2)(5*$M(t_1)$ +8* $M(t_2)$+ 5*$M(t_3)$)/9 ________________________________

Question 5 (18 marks)

The values for a function are provided in table below:

x	1	3	5	7	9
y	36	5	1	0	-5

(i) Estimate the derivative of this function at x=1 and x=5 using the formulas from the exam reference that will provide the most accurate results.

forward 3pt = - 22.2500 (at x=1 and h=2)
central 4pt = 0.0417 (at x=5 and h=2)

(ii) Assume that the above data contains noise/errors and that theory suggests a cubic relationship between x and y (i.e. that $y = ax^3 + bx^2 + cx + d$). Give Matlab code that uses an appropriate method to estimate the derivative at $x = 4$. An output statement has been provided.

```
p=polyfit(x,y,3);
pp=polyder(p);
derivativeat4=polyval(pp,4);
```

fprintf ('The derivate at x = 4 is %f\n', ______ derivativeat4__________);

(iii) Briefly explain how you would go about estimating the derivative at $x = 4$ if the data points were reliable and the relationship between x and y was unknown.

Choose a method to interpolate the data – cubic spline for example
- avoid high order polynomials

From this spline function – evaluate the derivative using central 4 difference at x=4 using a suitably small h (point spacing) to
get accuracy desired.

(iv) Using only the data points for $x = 1, 3,$ and 5, manually determine the polynomial that passes through these 3 points. Use your polynomial to estimate the value of the function at $x = 2$. Does your calculated estimate make sense given the data?

$$y(x) = \frac{(x-3)(x-5)\,36}{(1-3)(1-5)} + \frac{(x-1)(x-5)\,5}{(3-1)(3-5)} + \frac{(x-1)(x-3)\,1}{(5-1)(5-3)}$$

$$y(2) = \frac{(-1)(-3)\,36}{(-2)(-4)} + \frac{(1)(-3)\,5}{(2)(-2)} + \frac{(1)(-1)\,1}{(4)(2)} = 17.125$$

Yes it does make sense – the calculated result is between 36 and 5 about half way

(v)The derivative at some point for some function has been calculated using the central 2 point formula for three steps sizes with the following results

$h = 0.25$	derivative estimate = -13.7
$h = 0.5$	derivative estimate = -14.1
$h = 1.0$	derivative estimate = -15.9

Given these results calculate the best possible estimate of the derivative:

-15.9	-13.5	-13.572 <-- best result
-14.1	-13.567	
-13.7		

(vi) What is E_A for the derivative found in part (v)?

$$EA = \left|(-13.572-(-13.5))/ -13.572\right| = 0.0053$$

Question 6 (16 marks)

A first order ordinary differential equation is shown below.

$$\frac{dy}{dt} = y - 2\,t$$

Demonstrate your understanding of Euler's method by using it to complete the following table.

i	t_i	y_i	φ (slope)
0	0	3	3
1	0.5	4.5	3.5
2	1	6.25	4.25

ii) What would y_1 be if the midpoint method was used with a step size of 1 second?

slope at start = 3 ➔ mid point estimate of y = 3 (line 1 of above table)
slope at midpoint = 3.5 (from line 2 of above table)
y(1 sec) = 3 + 3.5 = 6.5

iii) What would y_1 be if the Heun's method (without iteration) was used with a step size of 0.5 seconds?

slope at start = 3 ➔ (line 1 of above table)
slope at end point estimate = 3.5 (line 2 in table)
average of the above = 3.25
y(0.5 sec) = 3 + 3.25/2 = 4.625

iv) Write a Matlab function that, given an some time t, returns the value of y at this time. Your function should accept the starting point to be used (t_0 and y_0) as well as the time of interest (t). You can (and should) make use the Matlab's ODE functions.

```
function [y_at_t] = exam1(t, t0, y0)
% comments
if t<t0
error(' in valid t')
end
dy=@(t,y) y-2*t;
[t,v] = ode45(dy,[t0 t],y0);
v_at_t = v(length(v));
end
```

Extra space