

**Question 1.** [4 points] Calculate

$$\text{a) } \int_{-2}^2 \frac{1}{y^2 - 9} dy \quad \text{b) } \int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{1 + \cos^2(x)} dx$$

**Solution.** a) We have

$$\frac{1}{y^2 - 9} = \frac{1}{(y - 3)(y + 3)} = \frac{a}{y - 3} + \frac{b}{y + 3} = \frac{(a + b)y + 3(a - b)}{(y - 3)(y + 3)},$$

with  $a + b = 0$  and  $3(a - b) = 1$ , which implies  $a = 1/6$  and  $b = -1/6$ .

$$\begin{aligned} \int_{-2}^2 \frac{1}{y^2 - 9} dy &= \frac{1}{6} \left( \int_{-2}^2 \frac{1}{y - 3} dy - \int_{-2}^2 \frac{1}{y + 3} dy \right) \\ &= \frac{1}{6} \left( [\ln|y - 3|]_{-2}^2 - [\ln|y + 3|]_{-2}^2 \right) \\ &= -\frac{1}{3} \ln(5) \end{aligned}$$

b) Substituting  $y = \cos(x)$ , we have  $dy = -\sin(x)dx$ .

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{1 + \cos^2(x)} dx &= \int_{\cos(-\pi/2)}^{\cos(\pi/2)} -\frac{1}{1 + y^2} dy \\ &= \int_0^0 -\frac{1}{1 + y^2} dy \\ &= [-\arctan(y)]_0^0 \\ &= 0 \end{aligned}$$

**Question 2.** [3 points] Solve the differential equation

$$\frac{dy}{dt} = \frac{6t \sin t}{y}$$

with initial condition  $y(0) = 5$ .

**Solution.** First we separate the differential equation, putting the  $y$ 's on the left and the  $t$ 's on the right, and add an integral sign to obtain

$$\int y dy = \int 6t \sin t dt.$$

The left-hand integral is simple:  $y^2/2$ . (We omit the  $+C$  with this integral.)

The right-hand integral requires integration by parts. Set  $u = t$ ,  $dv = \sin t dt$ , so that  $du = dt$ ,  $v = -\cos t$ . Then we obtain

$$\int 6t \sin t dt = 6 \int t \sin t dt$$

$$\begin{aligned}
&= 6 \left[ -t \cos t - \int (-\cos t) dt \right] \\
&= 6 [-t \cos t + \sin t] + C.
\end{aligned}$$

So the general solution to the differential equation is

$$y^2/2 = 6 [-t \cos t + \sin t] + C;$$

solving for  $y$  gives

$$y = \pm \sqrt{12(-t \cos t + \sin t) + C}. \quad (*)$$

(The constant  $C$  here is not the same as in the previous line. To be more accurate, you can write  $+2C$  or  $+D$  instead of  $+C$  in equation (\*), with  $D$  a different arbitrary constant.) This means that the various particular solutions are obtained by selecting one of  $\pm$  and a particular constant  $C$ .

In our case we're given the initial condition  $y(0) = 5$ , that is, when  $t$  is 0,  $y$  takes the value 5. Plugging in  $t = 0$  into equation (\*) gives

$$y(0) = \pm \sqrt{12(0 + 0) + C} = \pm \sqrt{C}$$

So  $\pm \sqrt{C} = 5$ . Clearly, to achieve the value of 5, we must select the positive square root. Squaring both sides gives  $C = 25$ . So the solution to our differential equation with initial condition is

$$\boxed{y = \sqrt{12(-t \cos t + \sin t) + 25}}.$$

**Question 3.** [4 points] Evaluate the integral

$$\int \frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12} dx.$$

**Solution.**  $P(x) = x^3 + 2x^2 - 18x + 2$ , and  $Q(x) = x^2 + x - 12$ . Then  $\deg(P) = 3 > 2 = \deg(Q)$ . So long division is necessary here. Using long division, we find that

$$x^3 + 2x^2 - 18x + 2 = (x + 1)(x^2 + x - 12) + (-7x + 14)$$

and

$$\frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12} = (x + 1) + \frac{-7x + 14}{x^2 + x - 12}$$

Again, we notice that  $x^2 + x - 12 = (x + 4)(x - 3)$ . We need to find out two numbers  $A$  and  $B$  so that

$$\frac{-7x + 14}{(x + 4)(x - 3)} = \frac{A}{x + 4} + \frac{B}{x - 3} = \frac{A(x - 3) + B(x + 4)}{(x + 4)(x - 3)} = \frac{(A + B)x + (-3A + 4B)}{(x + 4)(x - 3)}$$

Comparing the coefficients of the numerators, we obtain

$$A + B = -7,$$

$$-3A + 4B = 14.$$

Solving implies that  $A = -6$  and  $B = -1$ . Finally,

$$\int \frac{x^3 + 2x^2 - 18x + 2}{x^2 + x - 12} dx = \int \left[ (x+1) + \frac{-6}{x+4} + \frac{-1}{x-3} \right] dx = \frac{1}{2}x^2 + x - 6 \ln|x+4| - \ln|x-3| + C.$$

**Question 4.** [6 points] For each of the following improper integrals, determine whether it converges, and determine its value if it does.

$$\text{a) } \int_1^3 \frac{1}{t \ln t} dt \qquad \text{b) } \int_0^\infty \frac{3}{18 + 2t^2} dt \qquad \text{c) } \int_5^\infty \frac{\ln x}{x^2} dx$$

**Solution.** a) Using substitution, we have

$$\begin{aligned} u &= \ln t \\ \frac{du}{dt} &= \frac{1}{t} \\ dt &= t du \end{aligned}$$

Then

$$\begin{aligned} \int_1^3 \frac{dt}{t \ln t} &= \lim_{\epsilon \rightarrow 1^+} \int_{t=\epsilon}^{t=3} \frac{1}{u} du \\ &= \lim_{\epsilon \rightarrow 1^+} [\ln|u|]_{t=\epsilon}^{t=3} \\ &= \lim_{\epsilon \rightarrow 1^+} [\ln|\ln t|]_{\epsilon}^3 \\ &= \lim_{\epsilon \rightarrow 1^+} \ln|\ln 3| - \ln|\ln \epsilon| \\ &= \infty, \end{aligned}$$

since when  $\epsilon$  is close to (but larger than) 1,  $\ln \epsilon$  is close to, but larger than, 0, so  $|\ln \epsilon| = \ln \epsilon$  and  $\ln|\ln \epsilon|$  approaches  $-\infty$ . Thus the integral diverges.

**b) Solution 1:**

$$\begin{aligned} \int_0^\infty \frac{3}{18 + 2t^2} dt &= \lim_{T \rightarrow \infty} \int_0^T \frac{3}{18 + 2t^2} dt \\ &= \lim_{T \rightarrow \infty} \frac{3}{18} \int_0^T \frac{1}{1 + t^2/9} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{6} \int_0^T \frac{1}{1 + (t/3)^2} dt \end{aligned}$$

Let  $u = t/3$  so  $dt = 3du$ . Then

$$\int_0^\infty \frac{3}{18 + 2t^2} dt = \lim_{T \rightarrow \infty} \frac{1}{2} \int_{t=0}^{t=T} \frac{1}{1 + u^2} du$$

$$\begin{aligned}
&= \lim_{T \rightarrow \infty} \frac{1}{2} \arctan u \Big|_{t=0}^{t=T} \\
&= \lim_{T \rightarrow \infty} \frac{1}{2} \arctan(t/3) \Big|_0^T \\
&= \lim_{T \rightarrow \infty} \frac{1}{2} \arctan(T/3) - \frac{1}{2} \arctan(0) \\
&= \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{4}
\end{aligned}$$

**Solution 2:** This time we'll make use of the formula

$$\int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan\left(\frac{t}{a}\right) + C, \tag{*}$$

a slightly easier variant of the formula presented in class. Let's start with the indefinite integral, and start by pulling out constant factors:

$$\int \frac{3}{18 + 2t^2} dt = \frac{3}{2} \int \frac{dt}{9 + t^2} = \frac{3}{2} \int \frac{dt}{t^2 + 3^2} = \frac{1}{2} \arctan\left(\frac{t}{3}\right) + C,$$

where we used (\*) for the last equality. Putting in the limits for our improper integral gives:

$$\begin{aligned}
\int_0^\infty \frac{3}{18 + 2t^2} dt &= \lim_{T \rightarrow \infty} \int_0^T \frac{3}{18 + 2t^2} dt = \lim_{T \rightarrow \infty} \frac{1}{2} \arctan\left(\frac{t}{3}\right) \Big|_0^T \\
&= \frac{1}{2} \lim_{T \rightarrow \infty} (\arctan(T/3) - 0) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.
\end{aligned}$$

c)

$$\int_5^\infty \frac{\ln x}{x^2} dx = \lim_{T \rightarrow \infty} \int_5^T \frac{\ln x}{x^2} dx.$$

Using substitution, we have

$$\begin{aligned}
u &= \ln x \\
\frac{du}{dx} &= \frac{1}{x} \\
dx &= x du
\end{aligned}$$

Then

$$\int_5^\infty \frac{\ln x}{x^2} dx = \lim_{T \rightarrow \infty} \int_{x=5}^{x=T} \frac{u}{x} du$$

We need to resubstitute. Since  $u = \ln x$ , it follows that  $x = e^u$ . Thus

$$\int_5^\infty \frac{\ln x}{x^2} dx = \lim_{T \rightarrow \infty} \int_{x=5}^{x=T} ue^{-u} du.$$

Next we use integration by parts and let  $w = u$  and  $v' = e^{-u}$ . Thus

$$\begin{aligned} \int_5^\infty \frac{\ln x}{x^2} dx &= \lim_{T \rightarrow \infty} \left[ -ue^{-u} + \int e^{-u} du \right] \\ &= \lim_{T \rightarrow \infty} \left[ -ue^{-u} - e^{-u} \right]_{x=5}^{x=T} \\ &= \lim_{T \rightarrow \infty} \left[ -\ln x e^{-\ln x} - e^{-\ln x} \right]_5^T \\ &= \lim_{T \rightarrow \infty} \left[ -\frac{\ln x}{x} - \frac{1}{x} \right]_5^T \\ &= \lim_{T \rightarrow \infty} \left[ -\frac{\ln T}{T} - \frac{1}{T} \right] - \left[ -\frac{\ln 5}{5} - \frac{1}{5} \right] \\ &= \lim_{T \rightarrow \infty} \left[ -\frac{1/T}{1} - \frac{1}{T} \right] - \left[ -\frac{\ln 5}{5} - \frac{1}{5} \right] \quad (\text{using L'H\^opital's rule}) \\ &= [0 - 0] - \left[ -\frac{\ln 5}{5} - \frac{1}{5} \right] \\ &= \frac{\ln 5}{5} + \frac{1}{5} \end{aligned}$$

Therefore, the integral converges.

**Question 5.** [4 points] Define functions  $f$  and  $g$  by

$$f(x) = 3x^2 \qquad g = -3x + 6$$

- a) Show that  $f$  and  $g$  intersect at points  $x = -2$  and  $x = 1$ .
- b) Calculate the area between  $f$  and  $g$  in the interval  $-4 \leq x \leq 2$ .

**Solution.** a) The points of intersection of  $f$  and  $g$  satisfy

$$f(x) - g(x) = 3x^2 + 3x - 6 = 3(x^2 + x - 2) = 3(x - 1)(x + 2) = 0.$$

Thus  $x_1 = 1$  and  $x_2 = -2$  are the intersection points.

b) We have  $a < -2$  and  $b > 1$ . We need to calculate the area between  $f$  and  $g$  on the interval  $[a, b]$ . This is

$$\int_a^b |f(x) - g(x)| dx = \int_a^b |3x^2 + 3x - 6| dx = 3 \int_a^b |(x - 1)(x + 2)| dx.$$

The function  $(x - 1)(x + 2)$  is negative for  $-2 < x < 1$ . Thus

$$\int_a^b |f(x) - g(x)| dx$$

$$\begin{aligned}
&= 3 \left( \int_a^{-2} (x-1)(x+2)dx + \int_{-2}^1 -(x-1)(x+2)dx + \int_1^b (x-1)(x+2)dx \right) \\
&= 3 \left( \int_a^{-2} (x^2 + x - 2) dx - \int_{-2}^1 (x^2 + x - 2) dx + \int_1^b (x^2 + x - 2) dx \right) \\
&= 3 \left( \left( \frac{x^3}{3} + \frac{x^2}{2} - 2x \right) \Big|_a^{-2} - \left( \frac{x^3}{3} + \frac{x^2}{2} - 2x \right) \Big|_{-2}^1 + \left( \frac{x^3}{3} + \frac{x^2}{2} - 2x \right) \Big|_1^b \right) \\
&= 3 \left( 2 \left( -\frac{8}{3} + 2 + 4 \right) - \left( \frac{a^3}{3} + \frac{a^2}{2} - 2a \right) - 2 \left( \frac{1}{3} + \frac{1}{2} - 2 \right) + \left( \frac{b^3}{3} + \frac{b^2}{2} - 2b \right) \right) \\
&= 27 + 3 \left( -\frac{a^3}{3} - \frac{a^2}{2} + 2a + \frac{b^3}{3} + \frac{b^2}{2} - 2b \right).
\end{aligned}$$

Substituting the values for  $a$  and  $b$  into the expression, we obtain

$$\int_{-4}^2 |f(x) - g(x)|dx = \boxed{45}.$$

**Question 6.** [4 points] **a)** A very skinny 2m-long snake has density  $\rho(x)$  kg/m at a distance  $x$  from the head of the snake, where

$$\rho(x) = 2x - x^2.$$

What is the total mass of the snake?

**b)** A less skinny 2m-long snake's body is given by rotating the function

$$y = \frac{e^{-x/2}}{10}, \quad 0 \leq x \leq 2$$

about the  $x$ -axis (the units of  $y$  is also m). What is the volume of this snake?

**Solution.** **a)** The total mass of the snake is

$$\int_0^2 \rho(x)dx = \int_0^2 (2x - x^2)dx = \left( x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ kg.}$$

**b)** The volume of this snake is

$$\begin{aligned}
&\int_0^2 \pi [y(x)]^2 dx = \int_0^2 \pi \left[ \frac{e^{-x/2}}{10} \right]^2 dx \\
&= \int_0^2 \pi \frac{e^{-x}}{100} dx = \frac{\pi}{100} \int_0^2 e^{-x} dx = \frac{\pi}{100} (-e^{-x}) \Big|_0^2 = \frac{\pi}{100} (-e^{-2} + e^0) = \frac{\pi}{100} (1 - e^{-2}) \text{ m}^3.
\end{aligned}$$

**Question 7.** [5 points] Determine the average value of  $f(x) = x \ln(x)$  over the range  $0 \leq x \leq 2$ . (Hint: to calculate an indeterminate limit, rearrange and use L'Hôpital's rule.)

**Solution.** The average value of  $x \ln(x)$  over this interval is given by the quotient

$$\frac{\int_0^2 x \ln(x) dx}{2 - 0} = \frac{1}{2} \int_0^2 x \ln(x) dx. \quad (*)$$

This integral is improper because  $\ln(x)$  is undefined at  $x = 0$ ; that means it is given as a limit of definite integrals:

$$\begin{aligned} \int_0^2 x \ln(x) dx &= \lim_{a \rightarrow 0^+} \int_a^2 x \ln(x) dx \\ &= \lim_{a \rightarrow 0^+} \left( \frac{x^2}{2} \ln x \Big|_a^2 - \int_a^2 \frac{x^2/2}{x} dx \right) \\ &= \lim_{a \rightarrow 0^+} \left( \frac{x^2}{2} \ln x \Big|_a^2 - \int_a^2 \frac{x}{2} dx \right) \\ &= \lim_{a \rightarrow 0^+} \left( \frac{x^2}{2} \ln x - \frac{x^2}{4} \right) \Big|_a^2 \\ &= (2 \ln 2 - 1) - \lim_{a \rightarrow 0^+} \left( \frac{a^2}{2} \ln a - \frac{a^2}{4} \right) \\ &= (2 \ln 2 - 1) - \frac{1}{2} \lim_{a \rightarrow 0^+} a^2 \ln a, \end{aligned}$$

where we applied integration by parts with  $u = \ln x$ ,  $dv = x dx$ , and in the last line, recognized that as  $a$  approaches 0, so will  $a^2/4$ .

Evaluating the remaining limit requires L'Hôpital's rule. We write the expression as a fraction in the most convenient way so that we can apply L'Hôpital:

$$a^2 \ln a = \frac{\ln a}{a^{-2}}.$$

As  $a$  approaches 0 (from the positive side), both numerator and denominator become infinite — the numerator approaches  $-\infty$  and the denominator approaches  $\infty$  — so the condition of L'Hôpital's rule is met and we can conclude that

$$\lim_{a \rightarrow 0^+} \frac{\ln a}{a^{-2}} = \lim_{a \rightarrow 0^+} \frac{1/a}{-2a^{-3}} = \lim_{a \rightarrow 0^+} -\frac{a^2}{2} = 0,$$

where we first applied L'Hôpital and then simplified. So in the end,  $\lim_{a \rightarrow 0^+} a^2 \ln a = 0$ , so the improper integral has value  $2 \ln 2 - 1$ , and the average value of the function is given by equation (\*) as  $\boxed{(2 \ln 2 - 1)/2}$