Mar 28th

Ex 7.18 Find the Taylor series of e^x, centered about $x=0$.

$f(x) = e^x \longrightarrow f(0) = 1$
$f'(x) = e^x \rightarrow f'(0) = 1$
$f''(x) = e^x \rightarrow f''(0) = 1$
$\vdots$
$f^{(n)}(x) = e^x \longrightarrow f^{(n)}(0) = 1$

$\therefore C_n = \frac{f^{(n)}(a)}{n!} \xrightarrow{a=0} \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

So $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Ex 7.19 What is the Taylor series of e^{2x}, centered at $x = 0$?

$f(x) = e^{2x} \longrightarrow f(0) = 1$
$f'(x) = 2e^{2x} \longrightarrow f'(0) = 2$
$f''(x) = 2 \cdot 2 \cdot e^{2x} \longrightarrow f''(0) = 2^2$
$f'''(x) = 2 \cdot 2 \cdot 2 \cdot e^{2x} \rightarrow f'''(0) = 2^3$
$\vdots$
$f^{(n)}(x) = 2^n e^{2x} \longrightarrow f^{(n)}(0) = 2^n$

$\therefore C_n = \frac{f^n(0)}{n!} = \frac{2^n}{n!}$

$\therefore e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

OR/ $e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n}$

$$e^{(2x)} = \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^{n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} 2^{n} x^{n}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^{n}}{n!} x^{n}$$

Ex 7.20 Determine the Taylor series of e^{x} centered at $x=3$

From before

$f = e^{x} \rightarrow f(3) = e^{3}$
$f' = e^{x} \rightarrow f'(3) = e^{3}$
$f'' = e^{x} \rightarrow f''(3) = e^{3}$
$f''' = e^{x} \rightarrow f'''(3) = e^{3}$

$\therefore C_n = \frac{f^{(n)}(3)}{n!} = \frac{e^{3}}{n!}$

$\therefore \sum_{n=0}^{\infty} \frac{e^{3}}{n!} (x-3)^{n}$

VERY IMPORTANT!

① $C_n = \dfrac{f^{(n)}(a)}{n!}$

② $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

④ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

NOTE: $(\sin x)' = \cos x$ ... ???

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\frac{d}{dx}(\sin x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \cos x$

Note, WLOG: $\frac{-3x^2}{3!} = \frac{-\cancel{3}x^2}{\cancel{3}\cdot 2\cdot 1} = -\frac{x^2}{2\cdot 1} = \frac{-x^2}{2!}$

ALSO: $\frac{d}{dx}(\cos x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots$

$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = -\sin x$$

Ex $\boxed{7.2 \quad = 22\ ??}$

$f(x) = \ln(x-1)$, $a = 2$

Find $T(x-2)$.

$f(x) = \ln(x-1)$

$f'(x) = \frac{1}{x-1} = (x-1)^{-1}$

$f''(x) = \frac{-1}{(x-1)^2} = (-1)(x-1)^{-2}$

$f'''(x) = \frac{(-1)(-1)(2)}{(x-1)^3}$

$f^{(4)}(x) = \frac{(-1)(-1)(-1)(2)(3)}{(x-1)^4}$

$f^{(5)}(x) = \frac{(-1)(-1)(-1)(-1)(2)(3)(4)}{(x-1)^5}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n}$, $n \geq 1$

So, $f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{(2-1)^n} = (-1)^{n-1}(n-1)!$

$$C_n = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}(n-1)!}{n(n-1)!} = \frac{(-1)^{n-1}}{n}$$

valid for $n \geqslant 1$

@ $n=0$, $f(2) = 0 \quad \ln(2-1) = \ln(1) = 0$

So, $T(x-2) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-2)^n$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-2)^n$$

Ex 7.2 *23 $f(x) = \ln x$, $x = 2$

$$f = \ln x$$

$$f' = \frac{1}{x}$$

$$f'' = \frac{-1}{x^2}$$

$$f''' = \frac{(-1)(-1)2}{x^3}$$

$$f^{(4)} = \frac{(-1)(-1)(-1)\cdot 2 \cdot 3}{x^4}$$

$$\vdots$$

$$f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$\therefore f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$$

$$\therefore C_n = \frac{f^{(n)}(2)}{n!} = \frac{\frac{(-1)^{n-1}(n-1)!}{2^n}}{n!} = \frac{(-1)^{n-1}}{n 2^n}$$ ← obvious, $n \geq 1$

$$\therefore T(x-2) = \underbrace{\ln 2}_{n=0} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} (x-2)^n}_{n \geq 1}$$

Note: Solution in the book:

$$\ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1) 2^{n+1}} (x-2)^{n+1}$$

Def'n A Maclaurin Series is a special case of Taylor series, such that the series is always centered about $x=0$.

Ex Find the Maclaurin Series for $f(x)=\frac{1}{(x-1)^2}$

$$f(x)=\frac{1}{(x-1)^2}$$

$$f'(x)=\frac{(-1)\cdot 2}{(x-1)^3}$$

$$f''(x)=\frac{(-1)(-1)(2)(3)}{(x-1)^4}$$

$$f'''(x)=\frac{(-1)(-1)(-1)\cdot 2\cdot 3\cdot 4}{(x-1)^5}$$

$$\vdots$$

$$f^{(n)}(x)=\frac{(-1)^n\ (n+1)!}{(x-1)^{n+2}}$$

$$f^{n}(0)=\frac{(-1)^n(n+1)!}{(-1)^{n+2}}=\frac{(-1)^n\ (n+1)!}{(-1)^n\cdot(-1)^2}=(n+1)!$$

$$C_n=\frac{(n+1)!}{n!}=\frac{(n+1)\,n!}{n!}=n+1$$

$$M(x)=\sum_{n=0}^{\infty}(n+1)x^n$$