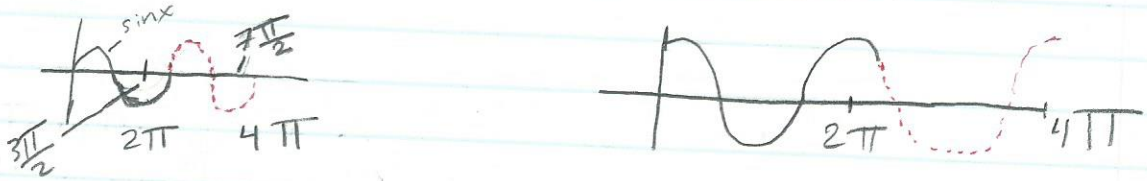
Apr 2^{nd}

Fourier Series

Def'n A f'n is periodic if for $T > 0$.

$$f(x+T) = f(x)$$

eg.



$$f\left(\frac{3\pi}{2}\right) = f\left(\frac{7\pi}{2}\right)$$

$$T = 2\pi$$

$f\left(\frac{3\pi}{2}\right)$ bcomes $f\left(\frac{3\pi}{2} + 2\pi\right) = f\left(\frac{7\pi}{2}\right)$

From the sketch, we see this is correct

* This works for BOTH $\sin x$ and $\cos x$

let $T = 2L$, where L is the half period.

THEN: Any $2L$-periodic f'n can be expressed as a fouries series:

$$f(x) = \frac{a}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow a_n = \int_{-1}^{1} (x^2+x) \cos(n\pi x)\, dx$$

$f(x)$		dx
x^2+x		$\cos(n\pi x)$
$2x+1$	$+$	$\frac{1}{n\pi} \sin(n\pi x)$
2	$-$	$\frac{-1}{n^2\pi^2} \cos(n\pi x)$
0	$+$	$\frac{-1}{n^3\pi^3} \sin(n\pi x)$

$$= \frac{x^2+x}{n\pi} \sin(n\pi x) \Big|_{-1}^{1}$$

$$+ \frac{2x+1}{n^2\pi^2} \cos(n\pi x) \Big|_{-1}^{1}$$

$$- \frac{2}{n^3\pi^3} \sin(n\pi x) \Big|_{-1}^{1}$$

NOTE:
$\sin(n\pi) = 0$
$\cos(n\pi) = (-1)^n$

$$= \frac{2(1)+1}{n^2\pi^2} \cos(n\pi) - \frac{2(-1)+1}{n^2\pi^2} \cos(-n\pi)$$

NOTE:
$\cos(-\alpha) = \cos(\alpha)$

$$\frac{3}{n^2\pi^2} \cos(n\pi) - \frac{-1}{n^2\pi^2} \cos(n\pi)$$

$$= \frac{4}{n^2\pi^2} \cos(n\pi) = \frac{4}{n^2\pi^2} (-1)^n$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow b_n = \int_{-1}^{1} (x^2 + x) \sin(n\pi x) dx$$

$x^2 + x$	$\oplus$ $\sin(n\pi x)$
$2x+1$	$\ominus$ $\frac{-1}{n\pi}\cos(n\pi x)$
2	$\oplus$ $\frac{-1}{n^2\pi^2}\sin(n\pi x)$
0	$\frac{1}{n^3\pi^3}\cos(n\pi x)$

$$= \frac{-x^2 - x}{n\pi}\cos(n\pi x)\Big|_{-1}^{1}$$

$$+ \frac{2x+1}{n^2\pi^2}\sin(n\pi x)\Big|_{-1}^{1}$$

$$+ \frac{2}{n^3\pi^3}\cos(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{-x^2 - x}{n\pi}\cos(n\pi x)\Big|_{-1}^{1}$$

$$+ \frac{2}{n^3\pi^3}\cos(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{-(1)^2 - (1)}{n\pi}\cos(n\pi) - \frac{-(-1)^2 - (-1)}{n\pi}\cos(-n\pi)$$

$$+ \frac{2}{n^3\pi^3}\cos(n\pi) - \frac{2}{n^3\pi^3}\cos(-n\pi)$$

$$= \frac{-2}{n\pi} \cos(n\pi) - \frac{0}{n\pi} \cos(-n\pi)$$

$$+ \frac{2}{n^3\pi^3} \cos(n\pi) - \frac{2}{n^3\pi^3} \cos(n\pi)$$

$$= \frac{-2}{n\pi} \cos(n\pi) - 0 + 0 = \frac{-2}{n\pi} (-1)^n$$

$$= \frac{(-1)(2)(-1)^n}{n\pi} = \frac{2}{n\pi} (-1)^{n+1}$$

So, $f(x) = \frac{2/3}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} (-1)^n \cos(n\pi x)$$

$$+ \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$$

ODD/EVEN FUNCTIONS

ODD/EVEN Extensions

FOURIER SINE SERIES

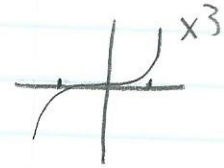
FOURIER COSINE SERIES

Definition A f'n is even if $f(\alpha) = f(-\alpha)$ eg. $\cos x$

A f'n is odd if $f(-\alpha) = -f(\alpha)$ eg. $\sin x$

Note: e^x is neither even, nor even.

Ex $f = x^3$ is odd since $f(-x) = -f(x)$



Test value ~~valume~~: $f(1) = 1$ opposite $f(-1) = -1$

Note: $f(x) = -f(-x)$

Ex $f(x) = x^2$ is even since $f(x) = f(-x)$

Test value:

$f(1) = 1$ same $f(-1) = 1$