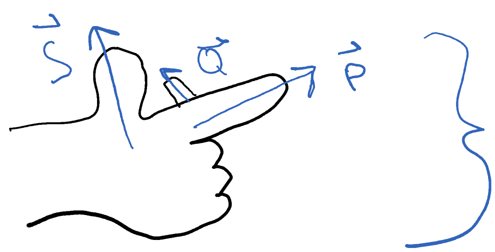
Chapter 3: Moment of a Force

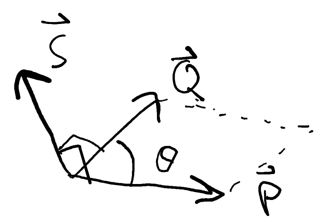
Before trying to solve problems involving rigid body, a fundamental concept to grasp is the moment of a force. This concept involves the possible rotation of an object about either a point or an axis. Let us begin by a few definitions:

Rigid Body: As stated before the rigid body is use to indicate an object that will not change its shape under external loads. Also, a rigid body has a geometry, which means that the point of application of the external forces on the object is important.

Vector product *(cross product):* The vector product of two vectors $\vec{P}$ and $\vec{Q}$, written $\vec{P} \times \vec{Q}$ or rarely $\vec{P} \wedge \vec{Q}$, is a vector $\vec{S}$ of magnitude $S = PQsin(\theta)$, theta being the angle $(0 \leq \theta \leq 180°)$ between P and Q, and of orientation given by the right hand rule.

Remember that $\vec{S}$ is always perpendicular to the plane made by $\vec{P}$ and $\vec{Q}$.





Right hand rule

Some properties of the cross-product:

1- $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$ or anticommutativity
2- $a\vec{P} \times b\vec{Q} = ab(\vec{P} \times \vec{Q})$
3- $\vec{P} \times \vec{P} = 0$

In rectangular coordinate, one can calculate the cross-product using the following formula:

$$\vec{P} \times \vec{Q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix} = \left(p_y q_z - p_z q_y\right)\vec{i} + \left(p_z q_x - p_x q_z\right)\vec{j} + \left(p_x q_y - p_y q_x\right)\vec{k}$$

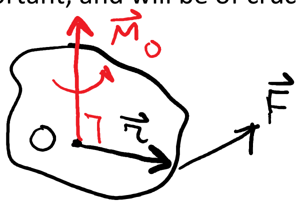
For smaller vectors, one can use the "shortcut": $\ominus$ $i\ j\ k\ i\ j$ $\oplus$

Moment of a force about a point: The tendency of a force to imply a rotating motion to a rigid body about a fixed point in space is called the Moment of a force about that point.

Unit: Nm Definition: about a point 0: $\overrightarrow{M_0} = \vec{r} \times \vec{F}$

The moment is a vector. Its direction indicates the orientation of the rotation, if rotation would be. The moment of a force is a function of the applied force and of the position of the force (the vector r). Which mean, the further away of a point on a rigid body you apply a force, the greater the moment of a force will be. For example, think of the knob of a door, which is located opposite to the hinges.

The cross product indicates that the moment will always be perpendicular to the direction of the applied force. This is important, and will be of crucial importance for moment about an axis.



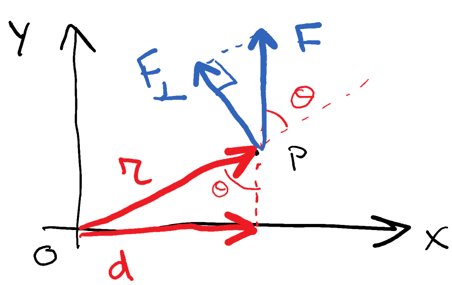
Special case: 2D

The moment of a force will always be in the z direction (perpendicular to the plane) in 2D. The orientation will either be going outward or inward. In this special case, one normally speaks of clockwise or anticlockwise rotation. It is of usage to identify the clockwise rotation by a negative sign and the anticlockwise rotation by a positive sign.

An interesting way to understand the moment of a force about a point is to use the notion of a lever. Everybody knows what a lever is. Whether one had to lift a heavy box or one watch cartoon on TV, the lever is the simplest machine one can imagine. The moment of a force is, in magnitude, the length between the point about which the moment is made and the external force, multiplied by the value of the force perpendicular to that length. This means that the moment is only effective if the force is applied perpendicular to the “rotation member” (just think of a door).

One can also take the distance perpendicular to the direction of the force. The result shall be the same.

$$M_o = Fd = rF_{\perp}$$



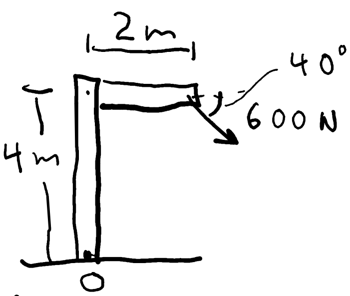
Before proceed to a simple example, let us introduce a Theorem to study body with multiple force. This theorem was proposed in a first time by Pierre Varignon (1654-1722) and shown in a more general case by ErnstMach in his famous book "The Science of Mechanics", 1883[1].

Varignon's Theorem (modern version)

The moment of a force about any point of a body is equal to the sum of the moments of the components of the force about the same point.

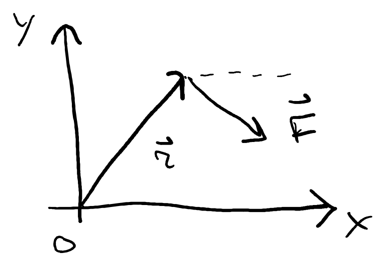
Example:

Find the moment of the 600N force about point O.



Solution:

FBD:



$$\vec{r} = (2\vec{i} + 4\vec{j})\ m$$

$$\vec{F} = (600\cos 40°\,\vec{i} - 600\sin 40°\,\vec{j})\ N$$

$$\therefore \vec{M}_0 = \vec{r} \times \vec{F} = (2\vec{i} + \overbrace{4\vec{j}) \times (459{,}6\vec{i} - 385{,}7\vec{j}}^{-(+\vec{k})})$$

$$\underbrace{\vec{j}) \times (459{,}6}_{-\vec{k}}$$

$$\boxed{\vec{M}_0 = -2609{,}8\,\vec{k}\ Nm} \text{ or } \vec{M}_0 = 2609{,}8\ Nm \text{ clock.}$$

To conclude with the moment, one needs another type of vector calculus: the scalar product.

The **scalar product** of two vectors is a product that leads to a scalar. It is denoted by $\vec{P} \cdot \vec{Q}$ and calculated by $PQcos(\theta)$, θ being the angle between the two vectors.

In rectangular coordinate, it is simply: $\vec{P} \cdot \vec{Q} = P_x Q_x + P_y Q_y + P_z Q_z$.

Now the reason one need to introduce this type of product is from the geometrical interpretation of the product. It is normally used to find the projection of a vector in the direction of another vector. Hence, in mechanics, it can be used to find the moment on object having some physical constrain like … a door!

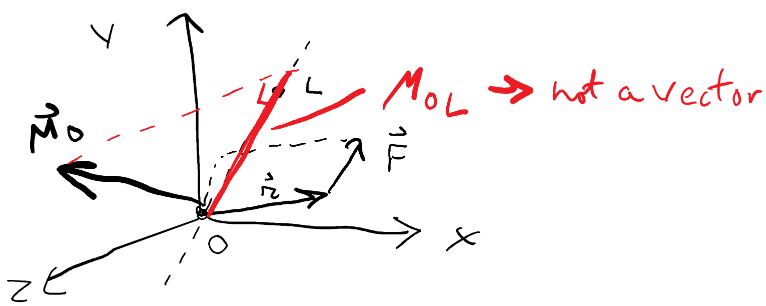
[1] This particular book played a strong influence in Einstein scientific life. It is among the few books that Albert Einstein cite in his few first papers on Special Relativity and Photoelectric Effect.

A door is fixed on its hinge. It only turns about a particular axis. If one push on the knob towards the ground (parallel to the hinges), nothing happen! There is no moment about the vertical axis.

Moment about an axis:

The moment of a force about an axis is given by the projection of the moment about a point on the axis and the direction of the axis. For example, if one wants the moment along the axis passing by O and L in the drawing below, then one has:

$$M_{OL} = \overrightarrow{\lambda_{OL}} \cdot \overrightarrow{M_o} = \overrightarrow{\lambda_{OL}} \cdot (\vec{r} \times \vec{F})$$



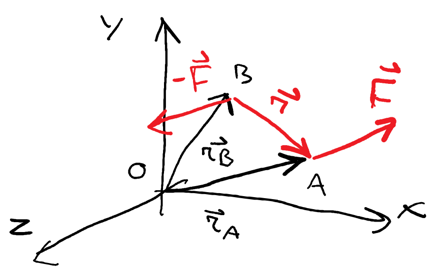
<u>**Couple and Moment of a Couple:**</u>

Definition:

Two forces having the same magnitude, parallel lines of action but opposite direction are called a <u>*couple*</u>.

A couple is useless to move an object in translation. The effect of the forces cancels each other. But, as a physical behavior, it is widely use because it gives a moment about an axis perpendicular to the plane made by the two forces. For example, think of a jar that you want to open. Your hand exerts two forces, on both side of the cover, in opposite direction. This allows the cover to turn and thus the jar to open.

Mathematically speaking, one can define the moment of a couple in the following manner:



According to the drawing, the total moment is the sum of the two moments about point o.

$$\overrightarrow{M_o} = \overrightarrow{r_A} \times \vec{F} + \overrightarrow{r_B} \times -\vec{F}$$

$$\overrightarrow{M_o} = (\overrightarrow{r_A} - \overrightarrow{r_B}) \times \vec{F} = \vec{r} \times \vec{F}$$

Hence, one need the relative position of the points of application of the two forces forming the couple, making a vector r and using the force vector of the end of the r vector thus formed.

Now, if one has to deal with real object, it may be interesting to know how to apply this couple to a fixed-dimension object. Therefore, one may have to choose either specific forces or distance to make a couple.

Two couples are said <u>equivalent</u> if they are located on parallel planes, and if they have moment of same magnitude and direction.

For more complex object, one may have to respect the position of the forces on the object, especially if not all the forces are forming couples. Therefore, a more complete definition of equivalence may be put forward:

Two systems of forces acting on a rigid body are said **equivalent** *if and only if*:

1- the sum of all the forces are equal, and,
2- Their resulting moments around a same point are the same.

The idea of the equivalent system is of prime importance to application. See either the notes in class or your textbook for more details.