MAT 1339 C Fall 2012 – Assignment #3
Due in class on Wednesday, November 7th

Name (print):________________________________

Student Number: SOLUTIONS

Instructions: Please print this assignment, enter your name and student number above, and answer all questions in the spaces provided. Your work must be shown to receive credit and all steps should be presented neatly, logically and clearly.

Question 1 Analyze the function $f(x) = \dfrac{2}{x^2 - x - 2}$ and sketch its graph.

(1) $f(x) = \dfrac{2}{(x-2)(x+1)}$ $\quad D = \mathbb{R} \setminus \{-1, 2\}$

(2) $\lim\limits_{x \to -1^-} f(x) = \infty$, $\lim\limits_{x \to -1^+} f(x) = -\infty \Rightarrow x = -1$ is VA

$\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to 2^+} f(x) = \infty \Rightarrow x = 2$ is VA

(3) Intercepts: $x = 0 \Rightarrow f(0) = -1 \Rightarrow (0, -1)$

$f(x) \neq 0$ for all $x \Rightarrow$ no x-intercept

(4) $\lim\limits_{x \to \infty} f(x) = 0$ $\quad \lim\limits_{x \to -\infty} f(x) = 0 \Rightarrow y = 0$ is HA

(5) $f'(x) = \dfrac{-2(2x-1)}{(x^2 - x - 2)}$, $f'(x) = 0$ if $x = \dfrac{1}{2}$

x	$-\infty$	-1		$\frac{1}{2}$		2	∞
y'	$+$	$\|\|$	$+$	0	$-$	$\|\|$	$-$
y	$\nearrow$	$\|\|$	$\nearrow$	L.Max	$\searrow$	$\|\|$	$\searrow$

Increasing: $(-\infty, -1) \cup (-1, \frac{1}{2})$

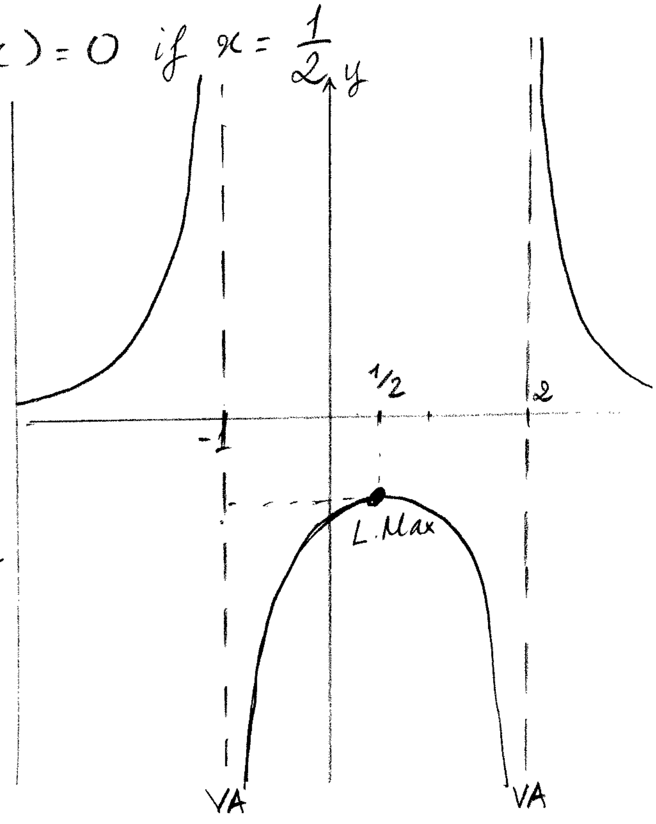
Decreasing: $(\frac{1}{2}, 2) \cup (2, \infty)$

Local Max: $(\frac{1}{2}, -\frac{8}{9})$

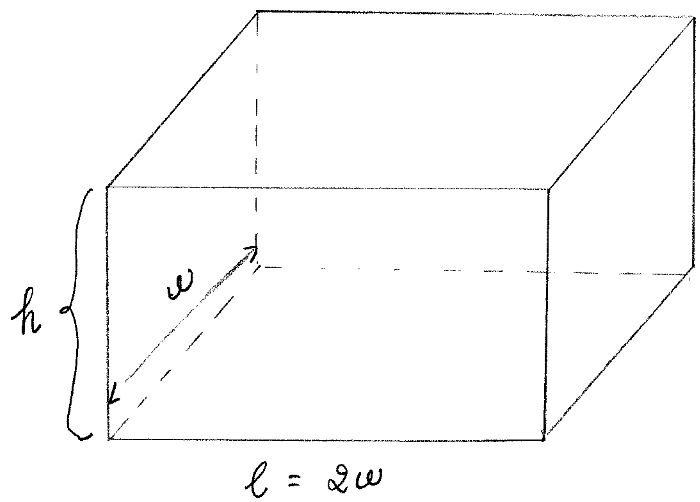
(6) $f''(x) = \dfrac{12(x^2 - x + 1)}{(x^2 - x - 2)^3} \neq 0$ for $\forall x$

Concave up: $(-\infty, -1) \cup (2, \infty)$

Concave down: $(-1, 2)$



Question 2 An open-top box is to be constructed so that its base is twice as long as it is wide and its volume is to be 2800 cm^3. Find the dimensions that will minimize the amount of cardboard required.



Let w be the width of the box,
l be the length of the box
h be the height of the box.
then $l = 2w$

The amount of cardboard required is equal to the surface area

$$A = \text{bottom} + 4 \text{ sides}$$
$$= w(2w) + 2wh + 2(2w)h$$
$$= 2w^2 + 2wh + 4wh$$
$$= 2w^2 + 6wh \longleftarrow \text{minimize}$$

Condition: $V = (\text{base})(\text{height}) = (2w)(w)(h)$
$$= 2w^2 h = 2800$$
$$\rightarrow h = \frac{2800}{2w^2} = \frac{1400}{w^2}$$

Bounds of w: $0 < w < \sqrt{1400}$

$$A(w) = 2w^2 + 6w\left(\frac{1400}{w^2}\right) = 2w^2 + \frac{8400}{w}$$
$$A'(w) = 4w - \frac{8400}{w^2} \Rightarrow A'(w) = 0$$
if $4w^3 = 8400$ or $w^3 = 2100$ or $w = \sqrt[3]{2100}$

Then, $w \approx 12.81$ cm, $l = 25.61$ cm, $h \approx 8.54$ cm.

$\left(A''(w) = 4 + \frac{16800}{w} > 0 \text{ for all } w > 0 \right)$

Question 3 Find the absolute maximum and minimum of $f(x) = x + e^{-x}$ on the interval $[-1, 3]$.

$f'(x) = 1 - e^{-x}$

$f'(x) = 0$ if $e^{-x} = 1$ or $e^{-x} = e^{0}$

$\Rightarrow$ $-x = 0$ then $x = 0$

$f(-1) = -1 + e^{-(-1)} = e - 1 \approx 1.72$

$f(0) = 0 + e^{0} = 1$

$f(3) = 3 + e^{-3} \approx 3.05$

Absolute max is 3.05 at $x = 3$

Absolute min is 1 at $x = 0$

Question 4 The radioactive element Unstablium has a half-life of 2.3 hours.
(a) How much of a 10 mg sample is left after 1 day?
(b) How long does it take for the sample to be reduced to 2 mg?

$m_0 = 10\,mg$, $T = 2.3$ hours

(a) $t = 1$ day $= 24$ hours

$m(t) = m_0 e^{-\frac{t \ln 2}{T}}$

$m(24) = 10\, e^{\frac{-24(\ln 2)}{2.3}} \approx 7.22 \times 10^{-3}\, mg$

(b) $m(t) = 2\,mg \Rightarrow t = ?$

$2 = 10\, e^{-\frac{t \ln 2}{2.3}} \Leftrightarrow e^{-\frac{t \ln 2}{2.3}} = 0.2$

$\ln\left(e^{-\frac{t \ln 2}{2.3}}\right) = \ln(0.2)$

$-\frac{t \ln 2}{2.3} = \ln(0.2)$

$t = \frac{-2.3 \ln(0.2)}{\ln 2} \approx 5.3$ (hours)

Question 5 Analyze the function $y = f(x) = x - \cos x$ on the interval $[0, 2\pi]$ and sketch its graph.

Considered domain $[0, 2\pi]$

no asymptotes, no symmetry

$x = 0$ they $f(0) = -1 \Rightarrow (0, -1)$

$f'(x) = 1 + \sin x$, $f'(x) = 0$ if $1 + \sin x = 0$

$\sin x = -1$ or $x = \frac{3\pi}{2}$

x	$-\infty$	0		$3\pi/2$		2π	∞
y'	//////		$+$	0	$-$	//////	
y	//////		$\nearrow$		$\nearrow$	//////	

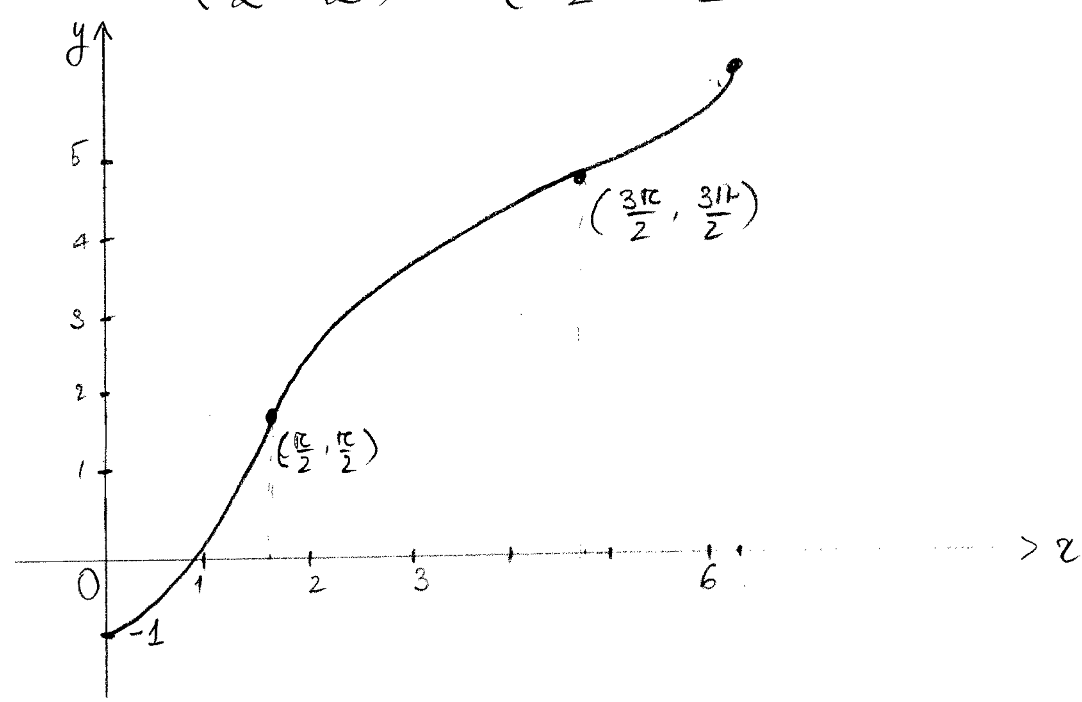
Increasing: $\left(0, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, 2\pi\right)$

$f''(x) = \cos x$, $f''(x) = 0$

if $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$

x	$-\infty$	0		$\frac{\pi}{2}$		$\frac{3\pi}{2}$		2π	∞
y''	//////		$+$	0	$-$	0	$+$	//////	
y	//////		up		down		up	//////	

IP: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$



Question 6 Analyze the function $y = f(x) = xe^x$ and sketch its graph.

$D = \mathbb{R}$

no VA

$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = 0 \Rightarrow y = 0$ is HA

$x = 0 \Rightarrow y = f(0) = 0 \Rightarrow (0,0)$

$f'(x) = (x+1)e^x$, $f'(x) = 0$ if $x + 1 = 0$

then $x = -1$

x	$-\infty$		-1		∞
y'		$-$	0	$+$	
y	0	$\searrow$	LMin	$\nearrow$	∞

Increasing : $(1, \infty)$, Decreasing $(-\infty, -1)$

CP is $x = -1$.

$f''(x) = (x+2)e^x$, $f''(x) = 0$ if $x + 2 = 0$

then $x = -2$

x	$-\infty$		-2		∞
y''		$-$	0	$+$	
y		down		up	

IP at $(-2, -2e^{-2})$