MAT 134A.F11 Final (soln)

3.19

1. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x \geq 0,\ y \geq 0 \text{ and } z \geq 0\}$. Then,

A. W is a subspace of $\mathbf{R}^3$.

B. $(0, 0, 0) \notin W$ and W is not closed under multiplication by scalars.

C. W is closed under addition but W is not closed under multiplication by scalars.

D. W is closed under addition and W is closed under multiplication by scalars.

E. W is not closed under addition but W is closed under multiplication by scalars.

F. None of the other statements is true.

Note that (I) $(0,0,0) \in W$

(II) W is closed under addition, but

(III) $(1,0,0) \in W$, but $(-1,0,0) = -1 \cdot (1,0,0) \notin W$,

so W is not closed under mult'n by scalars

2. If the augmented matrix $[A|b]$ of a system $Ax = b$ is row-equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which of the following statements is true?

Unique soln

$(5, -3, 1)$

A. The system is inconsistent

B. $X = (5,\ -2 - s,\ 1)$ is the solution for any value of s

C. $X = (5,\ -2,\ 1)$ is the unique solution of the system

D. $X = (5s,\ -2s,\ s)$ is a solution for any value of s

E. $X = (5t,\ -2 - s,\ s)$ is the solution for any value of s and t

F. $X = (5,\ -3,\ 1)$ is the unique solution to the system

3. For a non-homogeneous system of 12 equations in 15 unknowns, answer the following three questions:

- Can the system be inconsistent? Yes ①
- Can the system have infinitely many solutions? Yes ②
- Can the system have exactly one solution? No ③

$$\begin{array}{cc} & 15 \\ 12 & \left[\begin{array}{c|c} A & b \end{array}\right] \end{array} \qquad b \neq 0$$

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

① eg $A = 0$, $b = e_1$

② eg $A = b = 0$ (15 parameters!)

③ rank $A \le 12 < 15 = \#$ variables.
Hence if the system is consistent, it has at least 3 parameters in its general soln.

4. Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row of A^{-1} is $[-1\ -1\ 1]$.
C. The second row of A^{-1} is $[1\,2\ -1]$.
D. The first row of A^{-1} is $[2\ 0\ -1]$.
E. The second column of A^{-1} is $[0\ 2\ -1]^t$.
F. All of B, C, D, E are true.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & x \end{bmatrix}$. For which value(s) of x is A invertible?

A. $x \neq -1$
B. $x \neq 1$
C. $x \neq 0$
D. $x = -1$
E. $x = 1$
F. $x \neq \pm 1$

$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1+x \end{bmatrix}$, which has rank

3 iff $1+x \neq 0$ i.e. $x \neq -1$.

6. Find the value of t for which (4, 6, 3, t) belongs to span$\{(1,3,-4,1),(2,8,-5,-1),(-1,-5,0,2)\}$.

A. 0
B. 4
C. 7
D. 11
E. 13
F. 15

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 8 & -5 & 6 \\ -4 & -5 & 0 & 3 \\ 1 & -1 & 2 & t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 2 & -2 & -6 \\ 0 & 3 & -4 & 19 \\ 0 & -3 & 3 & t-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 37 \\ 0 & 0 & 0 & t-13 \end{array}\right],$$

and this is consistent $\Leftrightarrow$ $t = 13$

7. The dimension of $K = \{A \in \mathbf{M}_{33}(\mathbf{R}) \mid A = -A^t\}$ is:

A. 0
B. 2
C. 3
D. 4
E. 6
F. 9

$$K = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \middle| a,b,c \right\} = \mathrm{Span}\left\{ \overset{M_1}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{M_2}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}, \overset{M_3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}} \right\}.$$

Moreover $aM_1 + bM_2 + cM_3 = 0 \iff \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = 0 \iff$

$a = b = c = 0$ $\therefore$ $\{M_1, M_2, M_3\}$ is a basis for K so

$\dim K = 3.$

8. Which two of the following are subspaces of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1)f(2) = 0\}$ ✗
$T = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-x) = 2f(x), \forall x \in \mathbf{R}\}$ ✓
$U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) > 1\}$ ✗
$V = \{f \in \mathbf{F}(\mathbf{R}) \mid f(6) = 0\}$ ✓

A. T and U.
B. T and V.
C. S and T.
D. S and V.
E. S and U.
F. U and V.

- S is not a s.s. since $f(x) = x-1$ & $g(x) = x-2$ are elements of S but $f+g \notin S$ since $(f+g)(x) = 2x-3$ is not zero at either 1 or 2
- U is not a ss since $0 \notin U$.

Hence the answer must be T & V (and you can check that they are indeed ss. of $F(\mathbb{R})$.)

9. Let $A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. The dimension of the row space of A is:

A. 4
B. 3
C. 2
D. 1
E. 0
F. Infinite

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which

clearly has rank 3. (Recall dim row(A) = rank A)

10. If $C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and D is a $3 \times m$ matrix then the second row of the matrix CD is

A. not defined unless $m = 2$.
B. the same as the first row of D.
C. the same as the second row of D.
D. the sum of the first and the third row of D.
E. the sum of twice the second row of D and the third row of D.
F. twice the first row of D.

Write D in block row form.

$$D = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad r_i \in \mathbb{R}^m$$

$$CD = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2r_2 + r_3 \\ r_1 + r_3 \end{bmatrix}$$

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11. A dog named Lanso is advised by a nutritionist to take 5 units of vitamin A, 13 units of vitamin C and 23 units of vitamin D each day. Lanso can choose from the three brands I, II and III, and the amount of each vitamin in each capsule of the various brands is given below:

	I	II	III
vitamin A	1	1	0
vitamin C	2	1	1
vitamin D	4	3	1

Lanso is not capable of taking fractions of a capsule.

[2] a) After **defining your variables**, write down a system of equations in these variables, **together with all constraints**, that determine the possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for Lanso. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Let x = # pills of type I Lanso takes / day
y = " " II "
z = " " III "
1/3

Then $x, y, z \in \mathbb{Z}$, $x, y, z \geq 0$ and
1/3 1/3

$$x + y = 5$$ (to get vit A requirement)
$$2x + y + z = 13$$ (" C ")
$$\& \quad 4x + 3y + 1 = 23$$ (" D ").

3@1/3

[1] b) The reduced row-echelon form of the augmented matrix of the system in part (a) is:

$$\begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$x = 8 - s$$
$$y = -3 + s \quad , \quad s \in \mathbb{R}$$
$$z = s$$

1

11 c) Find all possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for Lanso.

[1]

$x, y, z \geq 0 \iff x \geq 0 \iff 8 - s \geq 0 \iff 8 \geq s$

$y \geq 0 \iff -3 + s \geq 0 \iff s \geq 3$

$z \geq 0 \iff s \geq 0.$

$\therefore 8 \geq s \geq 3, \quad s \in \mathbb{Z}.$

$\therefore \{ (8-s, -3+s, s) \mid s = 3, 4, 5, 6, 7, 8 \}$

$\frac{1}{2}$ - correct answer

$\frac{1}{2}$ — just^n.

11 d) Lanso has a tight budget. If the respective costs (in cents) per capsule of brands I, II and II are 4, 2 and 3, determine the choice which will minimize Lanso's total cost each day, and give this minimum cost per day.

[2]

Cost per day $= 4x + 2y + 3z$

$= 4(8-s) + 2(-3+s) + 3s$

$= 32 - 6 - 4s + 2s + 3s = 26 + s.$

This is minimized when $s = 3$ (since $s \geq 3$).

Thus Lanso should take 5 type I, 0 type II, 3 type III pills each day; cost $= 29$c/day

$\frac{1}{2} + \frac{1}{2}$ - correct answers + 1 just^n

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$\qquad\qquad v_1 \qquad v_2 \qquad\qquad v_3 \qquad\qquad v_4$

12. Let $W = \text{span}\{(1,0,0,1), (0,-1,-1,0), (0,0,0,1), (0,1,1,1)\} \subset \mathbf{R}^4$.

[$1\frac{1}{2}$] a) Find a basis of W which is a subset of the given spanning set.

[2] b) Find an orthogonal basis of W.

[$1\frac{1}{2}$] c) Find the best approximation to $(0,1,-1,1)$ in W. (This is $\text{proj}_W(0,1,-1,1)$)

[1] d) Extend your basis of W in (b) to a basis of $\mathbf{R}^4$.

a) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ so $\text{col}\,A = W$. Then $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(1) - any correct basis (1/2) - just...

Hence $\{v_1, v_2, v_3\}$ is a basis of W of the required form.

b) Note that $v_1 \cdot v_2 = 0$, so

(1) - {use of correct G.S. (or other...)

(1) - any correct orthog basis

$w_1 = v_1$

$w_2 = v_2$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2$$

$$= (0,0,0,1) - \frac{1}{2}(1,0,0,1) - \frac{(0)}{2}(0,-1,-1,0)$$

$$= \left(-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right).$$ Let $u_3 = 2w_3 = (-1,0,0,1)$

Hence $\{w_1, w_2, u_3\}$ is an orthogonal basis of W.

c) Now, $$\text{proj}_W(0,1,-1,1) = \frac{w_1 \cdot (0,1,-1,1)}{2} w_1 + \frac{w_2 \cdot (0,1,-1,1)}{2} w_2 + \frac{u_3 \cdot (0,1,-1,1)}{2} u_3$$

$$= \tfrac{1}{2}(1,0,0,1) + 0 w_2 + \tfrac{1}{2}(-1,0,0,1) = (0,0,0,1)$$

(1/2) - correct answer

(1) - using orthog proj correctly

d) We need w_4 so that

$$4 = \operatorname{rank}\begin{bmatrix} w_1 \\ w_2 \\ u_3 \\ w_4 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ & w_4 & & \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ & w_4 & & \end{bmatrix}.$$

Hence $w_4 = (0,0,1,0) = e_3$ will do.

Hence, $\{w_1, w_2, u_3, e_3\}$ is a basis of $\mathbb{R}^4$ of the required kind

$\frac{1}{2}$ - any correct extension

$\frac{1}{2}$ - justn

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13. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

[1] a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of A are 2 and -1.

[1] b) Find a basis of $E_2 = \{x \in \mathbf{R}^3 \mid Ax = 2x\}$.

[½] c) Find a basis of $E_{-1} = \{x \in \mathbf{R}^3 \mid Ax = -x\}$.

[1½] d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

[1] e) Find an invertible matrix $Q \neq P$ such that $Q^{-1}AQ = \tilde{D}$ is also diagonal, and give this diagonal matrix $\tilde{D}$.

a) $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 1+\lambda \\ 1 & -\lambda & 0 \\ 1 & 1 & -\lambda-1 \end{vmatrix} = (1+\lambda) \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 1 & -1 \end{vmatrix}$

$= (1+\lambda) \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1-\lambda & 2 & 0 \end{vmatrix} \overset{\text{col } 3}{=} (1+\lambda) \begin{vmatrix} 1 & -\lambda \\ 1-\lambda & 2 \end{vmatrix} = (1+\lambda)\{2 + \lambda - \lambda^2\}$

$= (1+\lambda)\{(2-\lambda)(1+\lambda)\}$

(1) - some correct work

$= 0 \iff \lambda = -1$ or $\lambda = 2$

b) $E_2 = \ker(A - 2I) = \ker \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$

$= \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{(1,1,1)\}$. Hence $\{\underset{v_1}{(1,1,1)}\}$ is a basis for E_2.

(1) - correct basis w/ justn

c) $E_{-1} = \ker(A + I) = \ker \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{\underset{v_2}{(-1,1,0)}, \underset{v_3}{(-1,0,1)}\}$

Hence $\{v_2, v_3\}$ is a basis of E_{-1}

(1½) - correct basis, justified

13(d) Set $P = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then

1/2 Correct assembly

$P \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$, which shows rank $P = 3$

hence P is invertible. If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then

1/2   1/2 consistent w P

(b) & (c) show that $AP = PD$ or $D = P^{-1}AP$.

13(e) Set $Q = [v_1\ v_3\ v_2]$, D as before. Then

col Q = col $P = \mathbb{R}^3$ so Q is invertible. Moreover, (b), (c)

show $Q^{-1}AQ = D$.

1/2 - correct (and distinct Q)

1/2 - consistent D

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14. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -1 & 4 & 3 \end{bmatrix}$$

(1) a) Find a basis for the column space $\operatorname{col}(A)$ of A.

(2) b) Give a complete geometric description of $\operatorname{col}(A)$.

(2) c) Find a basis for the kernel, $\ker T$, of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x) = Ax, \quad x \in \mathbf{R}^4.$$

(1) d) Compute $\dim(\ker T) + \dim(\operatorname{im} T)$.

a) $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -1 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} & s & & t \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

Hence $\{ (1,0,1), (2,2,4) \}$ is a basis of $\operatorname{col}(A)$

(1/2) any correct basis (1/2) justn?

b) Since $(1,0,1) \times (2,2,4) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 2 & 4 \end{vmatrix} = (-2, -2, 2)$,

col A is the $\underset{(1/2)}{\underline{\text{plane}}}$ $\underset{(1/2)}{\underline{\text{through}}}$ 0 in $\mathbb{R}^3$ with $\underset{(1/2)}{\underline{\text{normal}}}$ $\underset{(1/2)}{\underline{(1,1,-1)}}$.

consistent with (a)

c) We know $\ker T = \ker A$. By (a), $\ker A$

$= \operatorname{Span} \{ (1,1,0,0), (1,0,-1,1) \}$. Thus $\{ u_1, u_2 \}$ is a basis of $\ker T$.

with $u_1 = (1,1,0,0)$, $u_2 = (1,0,-1,1)$

(1/2) $\ker T = \ker A$

(1) any correct basis

(1/2) justn

d) We know by the conservation of dimension that

$\dim \ker T + \dim \operatorname{im} T = \dim \mathbb{R}^4 = 4$. Or, directly, $\operatorname{im} T = \operatorname{col} A$,

So $\dim \ker T + \dim \operatorname{im} T = \dim \ker A + \dim \operatorname{col} A = 2 + 2 = 4$.

(1/2) correct

(1/2) justn

15. a) Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

"$\det A \neq 0$", (So A is invertible)

one each in terms of:

(I) the columns of A

· are l.i

(· a basis of $\mathbb{R}^n$)

(· span $\mathbb{R}^n$)

1

(II) the reduced row-echelon form of A is I_n

(has n "ones" on the diagonal)

1

(III) the homogeneous system $Ax = 0$.

· has a unique soln $x = 0$

1

15b) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) If A and B are 2 by 2 matrices, then $\det(A+B)$ is always equal to $\det A + \det B$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A + \det B$ $= 0 + 0 = 0$, but $\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$.

— $\frac{1}{2}$ + $\frac{1}{2}$ ((i), (ii) & (iii)) —

correct just n.

$\boxed{\text{FALSE}}$

ii) If a 13×13 matrix A satisfies then $A^2 = 0$, then A is not invertible.

Suppose A were invertible. Then $A = A^{-1}(A^2) = A^{-1}(0) = 0$, but $\det 0 = 0$, so 0 is <u>not</u> invertible. Hence A is <u>not</u> invertible.

$\boxed{\text{TRUE}}$

iii) The columns of a 3×4 matrix are always linearly dependent.

The 4 columns of the matrix live in $\mathbb{R}^3$. But $\dim \mathbb{R}^3 = 3 < 4$. Hence the 4 columns are dependent.

$\boxed{\text{TRUE}}$

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16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

Suppose A is a 3 by 3 matrix such that $A = -A^t$ (i.e., A is antisymmetric). Prove that if $A \neq 0$, then the rank of A is exactly 2.

We know $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$.

Note $\det(A) = \det(A^t) = \det(-A) = (-1)^3 \det A = -\det A$, so $\det A = 0$. Thus rank $A < 3$. 1 The rank of $A \geq 1$ 1 since $A \neq 0$. It remains to show rank A cannot be 1.

Note rank $A = 1 \iff$ each row is a multiple of a fixed vector. In that case if $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, then $r_1 \times r_2 = 0$,

$r_1 \times r_3 = 0 = r_2 \times r_3$.

But $r_1 \times r_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & a & b \\ -a & 0 & c \end{vmatrix} = (*, -*, a^2)$, so $a = 0$.

Then $r_1 \times r_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & b \\ -b & 0 & 0 \end{vmatrix} = (*, -b^2, *)$, & $b = 0$

Thus $r_2 \times r_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & c \\ 0 & -c & 0 \end{vmatrix} = (c^2, 0, 0)$, so $c = 0$.

Thus rank $A = 1 \iff A = 0$. Since $A \neq 0$, rank $A = 2$.

+ 1 – some progress

4[14] + 1 – correct and well-written