MAT 1341A Fall 2011 Test 1 Solutions

1. Let $\mathcal{P}_2 = \{\, p \mid p(x) = a + bx + cx^2,$ where $a, b, c \in \mathbf{R}\}$ be the vector space of polynomial functions of degree at most 2, and consider the following subset of $\mathcal{P}_2$:

$$S = \{\overset{p}{x^2 - 1},\ \overset{q}{x^2 + 1},\ \overset{r}{x - 1},\ \overset{s}{x + 1}\}$$

Which of the following statements are true? (*) $p - q = -2 = r - s$ $\therefore$ S is l.d.

I. S is linearly dependent ✓
II. S is linearly independent ✗
III. S spans $\mathcal{P}_2$ ✓
IV. S is a basis of $\mathcal{P}_2$ impossible by I.

Since $1 \in \operatorname{span} S$ by (*)
$x = r + 1 \in \operatorname{span} S$, and
$x^2 = p + 1 \in \operatorname{span} S$.
Hence $\operatorname{span} S = \operatorname{span}\{1, x, x^2\} = \mathcal{P}_2$.
Thus S spans $\mathcal{P}_2$

A. (I) and (II) is impossible
B. (I) and (III)
C. (II) and (IV)
D. (II) and (III)
E. (I), (III) and (IV)
F. (III) and (IV)

2. A vector space V has dimension 15 and W is a subspace of V in which $\{v_1, \ldots, v_8\}$ is a spanning set. Which of the following statements are always true?

I. $\dim W < 8$ not necessarily: $\{v_1, \ldots, v_8\}$ could be l.i. ✗
II. $\dim W \leq 15$ ✓ since $\dim W \leq \dim V = 15$ ✓
III. $\dim W \leq 7$ This is the same statement as I! ✗
IV. Any linearly independent set in W has no more than 8 vectors in it. ✓

True, since size l.i. set $\leq$ size of a spanning set

A. (I) and (II)
B. (I) and (III)
C. (II) and (IV)
D. (II) and (III)
E. (I), (III) and (IV)
F. (III) and (IV)

3. Which of the following are subspaces of $\mathbf{R}^3$?

(I) $\{(x,\ y,\ z) \mid x - 2y = 0\}$ This is a plane through the origin ✓
(II) $\{(x,\ y,\ z) \mid xyz = 0\}$ This is not closed under addition; see below
(III) $\{(x,\ y,\ z) \mid y = 2z\}$ This is a plane through the origin ✓
(IV) $\{(x,\ y,\ z) \mid x = y + 3 = z\}$ This does not contain zero.

A. (I) and (II)
B. (I) and (III)
C. (II) and (IV)
D. (II) and (III)
E. (I), (III) and (IV)
F. (III) and (IV)

Note: for II,

$u = (1,0,0) \in \mathrm{II}$, and

$v = (0,1,1) \in \mathrm{II}$, but

$u+v = (1,1,1) \notin \mathrm{II}$.

4. Let $\mathbf{F}([-1,1]) = \{f \mid f : [-1,1] \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $[-1,1]$. Recall that the zero of $\mathbf{F}[-1,1]$ is the function that has the value 0 for all $x \in [-1,1]$.

Define three functions in $\mathbf{F}([-1,1])$ by

$$f(x) = x, \qquad g(x) = 1 - 2x + x^2, \quad \text{and} \quad h(x) = 1 + x^3.$$

Now let $W = \text{span}\{f, g, h\}$.

a) Show that f, g and h are linearly independent.

b) Find a basis for W and the dimension of W.

c) If $k(x) = 2 + x^2 + x^3$ show that $k \in W$.

d) What is the dimension of the subspace $Y = \text{span}\{f, g, h, k\}$?

a) Suppose $af + bg + ch = 0$, so $ax + b(1-2x+x^2) + c(1+x^3) = 0$, $\forall x \in [-1,1]$.

At $x=-1$ we obtain $-a + 4b = 0$ ①

$x = 0$ " $b + c = 0$ ②

$x = 1$ " $a + 2c = 0$ ③

① + ③ yields $4b + 2c = 0$, which together with ② yields $b = c = 0$, which by ① implies $a = b = c = 0$. Hence $\{f, g, h\}$ is l.i.

b) Since $\{f, g, h\}$ spans W by defn, and by (a) is l.i., $\{f, g, h\}$ is a basis for W, so $\dim W = 3$

c) Note $2 + x^2 + x^3 = 2f(x) + g(x) + h(x) \in W \therefore k \in W$

d) By (c), $k \in W$ so $Y = \text{span}\{f, g, h, k\}$
$= \text{span}\{f, g, h\}$
$= W.$

Thus $\dim Y = \dim W = 3$

5. Suppose $w = (0,2,1)$ and we define subspaces of $\mathbf{R}^3$ by

$$W = \{v \in \mathbf{R}^3 \mid v \times w = 0\}$$
$$U = \{v \in \mathbf{R}^3 \mid v \cdot w = 0\}$$

a) Show that $(0,2,1) \times (x,y,z) = (2z - y, x, -2x)$.

b) Use (a) to first find a spanning set for W, and then find a basis B for W.

c) Give a basis for U.

d) Extend your basis B to a basis of $\mathbf{R}^3$.

e) Give complete geometric descriptions of W and U.

a) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 1 \\ x & y & z \end{vmatrix} = (2z-y, -(-x), -2x)$, as required

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b) By (a) $W = \{(x,y,z) \mid 2z-y = 0 = x = -2x\}$
$= \{(0,2z,z) \mid z \in \mathbb{R}\} = \text{span}\{(0,2,1)\} = \text{span}\{w\}$.

Thus $\{w\}$ spans W. Moreover, $w \neq 0$ so $\{w\}$ is l.i. and hence is a basis for W. So $B = \{w\}$ is a basis for W

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c) $U = \{(x,y,z) \mid 2y + z = 0\} = \{(x,y,-2y) \mid x,y \in \mathbb{R}\}$
$= \text{span}\{\underset{u_1}{(1,0,0)}, \underset{u_2}{(0,1,-2)}\}$. (*) Since neither of u_1 nor u_2 is a multiple of the other, $\{u_1,u_2\}$ is l.i. By (*), " $\{u_1,u_2\}$ spans U and so $\{u_1,u_2\}$ is a basis of U.

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d) We claim that $\{u_1,u_2,w\}$ is a basis for $\mathbb{R}^3$. It suffices, because $\dim \mathbb{R}^3 = 3$ to show that $w \notin \text{Span}\{u_1,u_2\} = U$: But $w \cdot w = 5 \neq 0$, so $w \notin \text{span}\{u_1,u_2\}$. As $\{u_1,u_2\}$ is l.i, $\{u_1,u_2,w\}$ is too, and it has $3 = \dim \mathbb{R}^3$ vectors. $\{u_1,u_2,w\}$ is a basis for $\mathbb{R}^3$ that contains B.

e) From (a) W is the line through O with dir^n w, and from (b), U is plane through O with normal $(0,2,1) = w$.

6. Let u, v and w be vectors in a vector space V.

a) State carefully what "$\{u, v\}$ is linearly independent" means. (i.e., give the definition.)

$\{u, v\}$ is l.i. $\Leftrightarrow$ $(au + bv = 0$ for scalars $a, b \Rightarrow a = b = 0)$.

Now suppose

$\star$ $\{u, v\}$ is linearly independent and that $\{u, v, w\}$ is linearly *dependent.*

Under the above assumptions, either show that the statement in (b) and (c), is always true, that is, for any V, and for any vectors u, v and w satisfying $\star$, or give a counterexample (with $V = \mathbf{R}^2$ or $V = \mathbf{R}^3$) to show that the statement in (b) and (c) isn't always true.

b) $w \in \text{span}\{u, v\}$. This is TRUE, since otherwise $\{u, v, w\}$ would be l.i. by a theorem from class.

c) $u \in \text{span}\{v, w\}$. FALSE. Let $u = (1,0)$, $v = (0,1)$ and $w = (0,0)$.

Then $(1,0) \notin \text{span}\{(0,1),(0,0)\} = \text{span}\{(0,1)\}$ because $(1,0)$ & $(0,1)$ are l.i..

7. [Bonus] Suppose $\{u, v, w\}$ are three vectors in $\mathbf{R}^3$ such that $u \cdot v \times w = 2$. Prove carefully that $\{u, v, w\}$ is a basis of $\mathbf{R}^3$. *(A geometric argument involving 'volume' is not sufficient.)*

Since $u \cdot v \times w = w \cdot u \times v = v \cdot w \times u = 2$ we show that $u \notin \text{span}\{v, w\}$, and this will by the symmetry noted, show that $v \notin \text{Span}\{u, w\}$ and $w \in \text{span}\{u, v\}$. Thus $\{u, v, w\}$ will be l.i., and since $\dim \mathbb{R}^3 = 3$, $\{u, v, w\}$ will be a basis of $\mathbb{R}^3$.

So suppose $u = av + bw$ for scalars $a, b \in \mathbb{R}$.

Then
$$u \cdot (v \times w) - (av + bw) \cdot (v \times w)$$
$$= a(v \cdot v \times w) + b(w \cdot v \times w)$$
$$= a \cdot 0 + b \cdot 0$$
$$= 0, \text{ a contradiction.}$$

Hence $u \notin \text{Span}\{v, w\}$ and by the argument above, this shows that $\{u, v, w\}$ is a basis for $\mathbb{R}^3$.