COMP 232 Mathematics for Computer Science
Winter 2014
Midterm Exam

Name: ____________________________ Total Points:

ID: ____________________________ __________ / 59

Instructions. This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet. Use pen, not pencil. Do not detach any pages from this exam!

($3^{\text{pts}}_{\text{ea.}}$) 1. We know that $\{\wedge, \neg\}$ forms a functionally complete set of operators, meaning that any other operator can be defined in terms of $\{\wedge, \neg\}$ only, for example

9 pts

$$p \vee q =_{\mathsf{def}} \neg(\neg p \wedge \neg q)$$
$$p \to q =_{\mathsf{def}} \neg(p \wedge \neg q)$$

The Shaffer stroke $\uparrow$ is a binary operator that has the following truth table:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Show that the Shaffer stroke by itself is functionally complete, by defining in the space below, the following operators by using the Shaffer stroke only:

(a) $\neg p =_{\mathsf{def}} p \uparrow p$

(b) $p \wedge q =_{\mathsf{def}} (p \uparrow q) \uparrow (p \uparrow q)$

(c) $p \vee q =_{\mathsf{def}} (p \uparrow p) \uparrow (q \uparrow q)$

9 pts

(2$^{\text{pts}}_{\text{ea.}}$) 2. For each of the following propositional sentences, state whether or not it is a tautology.

You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

10 pts

(a) $((p \vee q) \wedge (q \vee r)) \leftrightarrow (q \vee r)$

☐ Tautology ☑ Not tautology ☐ Don't know!

(b) $(p \to q) \to (q \to p)$

☐ Tautology ☑ Not a tautology ☐ Don't know!

(c) $((p \to q) \wedge (q \to p)) \leftrightarrow (p \leftrightarrow q)$

☑ Tautology ☐ Not a tautology ☐ Don't know!

(d) $((p \to q) \wedge (\neg r \to \neg q)) \to (p \to (q \wedge r))$

☐ Not a tautology ☑ Tautology ☐ Don't know!

(e) $(p \to (q \to r)) \leftrightarrow ((p \to q) \to r)$

☐ Tautology ☑ Not a tautology ☐ Don't know!

10 pts

(6 pts ea.) 3. Here you are to prove propositional equivalences using the laws in the handout.

12 pts

(a) In the table below, construct a proof of the equivalence

$$(r \vee p) \to (r \vee q) \equiv r \vee (p \to q)$$

Step			Law applied
$(r \vee p) \to (r \vee q)$	$\equiv$	$\neg(r \vee p) \vee (r \vee q)$	Implication
	$\equiv$	$\big((\neg r) \wedge (\neg p)\big) \vee (r \vee q)$	de Morgan
	$\equiv$	$\Big(\big((\neg r) \wedge (\neg p)\big) \vee r\Big) \vee q$	Associativity
	$\equiv$	$\Big(r \vee \big((\neg r) \wedge (\neg p)\big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\big(r \vee (\neg r)\big) \wedge \big(r \vee (\neg p)\big)\Big) \vee q$	Distributivity
	$\equiv$	$\Big(\big(r \vee (\neg p)\big) \wedge \big(r \vee (\neg r)\big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\big(r \vee (\neg p)\big) \wedge T\Big) \vee q$	Excluded middle
	$\equiv$	$\big(r \vee (\neg p)\big) \vee q$	Identity
	$\equiv$	$r \vee \big((\neg p) \vee q\big)$	Associativity
	$\equiv$	$r \vee (p \to q)$	Implication

(b) In the table below, construct a proof of the equivalence

$$\neg(p \to q) \to p \equiv True$$

Step			Law applied
$\neg(p \to q) \to p$	$\equiv$	$\neg\neg(p \to q) \vee p$	Implication
	$\equiv$	$(p \to q) \vee p$	Double negation
	$\equiv$	$(\neg p \vee q) \vee p$	Implication
	$\equiv$	$(q \vee \neg p) \vee p$	Commutativity
	$\equiv$	$q \vee (\neg p \vee p)$	Associativity
	$\equiv$	$q \vee True$	Excluded middle
	$\equiv$	$True$	Domination

12 pts

(2$^{\text{pts}}_{\text{ea.}}$) 4. Let the universe of discourse be $\mathbb{Z}^+$, the set $\{1, 2, 3, \ldots\}$ of positive integers. Let $P(x, y, z)$ denote the statement "z is a multiple of $x+y$" or, equivalently, "There is a positive integer q, such that $z = (x + y)q$"

8 pts

What is the truth value of each of the following? You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

(a) $\forall x \exists y \exists z P(x, y, z)$

- [x] True
- [ ] False
- [ ] Don't know!

(b) $\forall y \forall z \exists x P(x, y, z)$

- [ ] True
- [x] False
- [ ] Don't know!

(c) $\forall x \forall y \exists z P(x, y, z)$

- [ ] False
- [x] True
- [ ] Don't know!

(d) $\forall z \exists x \exists y P(x, y, z)$

- [x] False
- [ ] True
- [ ] Don't know!

8 pts

(4pts) 5. The negation of the statement $\forall x \neg \forall y \exists z (P(x,z) \wedge Q(z,y))$ is

4 pts

- [ ] $\forall x \exists y \forall z (\neg P(x,z) \vee \neg Q(z,y))$
- [x] $\exists x \forall y \exists z (P(x,z) \wedge Q(z,y))$
- [ ] $\forall x \exists y \forall z (\neg P(x,z) \wedge \neg Q(z,y))$
- [ ] $\exists x \exists y \forall z (P(x,z) \vee \neg Q(z,y))$
- [ ] $\forall x \forall y \exists z (\neg P(x,z) \wedge \neg Q(z,y))$

(4pts) 6. The proposition $(p \leftrightarrow r) \rightarrow (q \leftrightarrow r)$ is equivalent to

4 pts

- [ ] $((\neg p \vee r) \wedge (p \vee \neg r)) \wedge ((\neg q \vee r) \wedge (q \vee \neg r))$
- [ ] $((\neg p \vee r) \wedge (p \vee \neg r)) \vee ((\neg q \vee r) \wedge (q \vee \neg r))$
- [x] $\neg((\neg p \vee r) \wedge (p \vee \neg r)) \vee ((\neg q \vee r) \wedge (q \vee \neg r))$
- [ ] $((\neg p \vee r) \wedge (p \vee \neg r)) \vee \neg((\neg q \vee r) \wedge (q \vee \neg r))$

(4pts) 7. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but are not overconfident."

4 pts

- [x] "If you lose the game then you don't know the rules or you are overconfident."
- [ ] "A necessary condition that you know the rules or you are not overconfident is that you win the game."
- [ ] "If you don't know the rules or are overconfident then you lose the game."
- [ ] "If you don't know the rules and are overconfident then you win the game."
- [ ] "A sufficient condition that you win the game is that you know the rules or you are not overconfident."

12 pts

(4pts) 8. Let $P(x, y)$ mean "x loves y." Which of the following expresses the proposition "Everybody loves somebody who doesn't love anybody."

4 pts

- [x] $\forall x \exists y\ (P(x, y) \wedge \forall z(\neg P(y, z)))$
- [ ] $\forall x \exists y \forall z\ (P(x, y) \wedge \neg P(z, y))$.
- [ ] $\forall x \forall y\ (P(x, y) \wedge \exists z(\neg P(y, z)))$
- [ ] $\forall x \exists y \exists z\ (P(x, y) \wedge \neg P(z, y))$.

(4pts) 9. Let the Universe of Discourse be $\mathbb{Z}$. Consider the assertion

4 pts

$$\exists x\Big(P(x) \wedge Q(x)\Big) \equiv \Big(\exists x\big(P(x)\big)\Big) \wedge \Big(\exists x\big(Q(x)\big)\Big)$$

Which of the following statements correctly describes the assertion?

- [ ] The assertion is false. As a counterexample, let $P(x)$ mean "x is divisible by 6," and $Q(x)$ mean "x is divisible by 3."
- [ ] The assertion is true. The proof follows from the distributive laws for $\wedge$
- [x] The assertion is false. For a counterexample, let $P(x)$ mean "" $x < 0$," and $Q(x)$ mean "$x \geq 0$."
- [ ] The assertion is false. For a counterexample, let $P(x)$ mean "''$x < 0$ is a square" and $Q(x)$ mean "x is odd."
- [ ] The assertion is true. To see why, let $P(x)$ mean "x is divisible by 6," and $Q(x)$ mean "x is divisible by 3." If $x = 6$, then x is divisible by both 3 and 6, so both side of the equivalence have the same truth value for this x.

—... End of Exam ...—

8 pts