1. Which of the following sets in $\mathbf{R}^4$ are closed under the usual operation in $\mathbf{R}^4$ of multiplication by scalars ?

$S = \{(a,\ b,\ c,\ d) \in \mathbf{R}^4 |\ ab = 0 \text{ and } d = 0\}$
$T = \{(a,\ b,\ c,\ d) \in \mathbf{R}^4 |\ a + b = 1 \text{ and } c = d\}$ $(1,0,0,0) \in T$, but $2.(1,0,0,0) \notin T$
$U = \{(a,\ b,\ c,\ d) \in \mathbf{R}^4 |\ b \geq 0 \text{ and } c \leq 0\}$ $v = (0,1,0,0) \in U$ but $-1.v \in T$
$V = \{(a,\ b,\ c,\ d) \in \mathbf{R}^4 |\ a + b - c + 2d = 0\}$

• S is closed under mult^n by scalars: if $v = (a, b, c, 0) \in S$, and $k \in \mathbb{R}$, $kv = (ka, kb, kc, 0)$. Then $(ka)(kb) = k^2ab = 0$, and "d" $= 0$. Thus $kv \in S$.

• V is closed under mult^n by scalars: If $v = (a,b,c,d) \in V$ and $k \in \mathbb{R}$, $kv = (ka, kb, kc, kd)$ and $ka + kb - kc + 2kd = k(a+b-c+2d) = k.0 = 0$. Hence $kv \in V$.

ANSWER $\boxed{S \text{ and } V}$

2. Which of the following are subspaces of $\mathbf{F}[-1, 1] = \{f \mid f: [-1, 1] \to \mathbf{R}\}$?

$X = \{f \in \mathbf{F}[-1, 1] \mid f(-1) = f(1)\}$
$Y = \{f \in \mathbf{F}[-1, 1] \mid f(0) = -1\}$ $0 \notin Y$
$Z = \{f \in \mathbf{F}[-1, 1] \mid f(x) = f(y),\ \forall x, y \in [-1, 1]\}$
$W = \{f \in \mathbf{F}[-1, 1] \mid f(-1) \leq 0\}$ If $f(x) = -2$, $\forall x \in [-1,1]$, then $f \in W$ but $-f \notin W$ since $-f(-1) = 2$.

X is a subspace: 1) $0(-1) = 0 = 0(1)$ $\therefore$ $0 \in X$
2) If $f, g \in X$, then $(f+g)(-1) = f(-1) + g(-1) = f(1) + g(1) = (f+g)(-1)$. So $f+g \in X$ and thus X is closed under add^n.
3) If $f \in X$ and $k \in \mathbb{R}$, $(kf)(-1) = k.f(-1) = k.f(1) = (kf)(1)$, so $kf \in X$. Hence X is closed under mult^n by scalars

Z is a subspace 1) $0(x) = 0 = 0(y)$, $\forall x,y \in [-1,1]$, so $0 \in Z$
2) If $f, g \in Z$, then $(f+g)(x) = f(x) + g(x) = f(y) + g(y) = (f+g)(y)$, $\forall x,y$ in $[-1,1]$, so $f+g \in Z$.

Note: this can also be done by noticing that Z consists of all constant functions on $[-1,1]$.

3) If $f \in Z$ and $k \in \mathbb{R}$, $(kf)(x) = k f(x) = k f(y) = (kf)(y)$, $\forall x,y \in [-1,1]$, so $kf \in Z$.

ANSWER $\boxed{X, Z.}$

3. Which of $U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbf{R}\}$, $V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbf{R}\}$ and $W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$ are subspaces of $\mathbf{R}^3$?

$U = \{x(1,1,1) + y(-1,1,-1) \mid x,y \in \mathbb{R}\} = \text{span}\{(1,1,1), (-1,1,-1)\}$. Hence U is a subspace of $\mathbb{R}^3$

Similarly, $V = \text{span}\{(1,0,0), (0,1,-1)\}$ is a subspace of $\mathbb{R}^3$.

Note that $w = (4,0,2) \in W$ but $2 \cdot w = (8,0,4) \notin W$. Hence W is not closed under mult'n by scalars and so is not a subspace of $\mathbb{R}^3$.

ANSWER $\boxed{U \text{ and } V}$

4. Which of the following are spanning sets for the subspace U of $\mathbf{R}^3$ defined by

$$U = \{(x, y, z) \mid x - y + z = 0\}?$$

Note $u = (x,y,z) \in U \iff x = y - z$ (y, z arbitrary)

$\iff u = (y-z, y, z)$

$= y(1,1,0) + z(-1,0,1)$

Thus $U = \text{span}\{(1,1,0), (-1,0,1)\}$, so C is certainly correct.

A. $\{(1,\ 1,\ 0)\}$

B. $\{(1,\ 1,\ 0),\ (0,\ 0,\ 0)\}$

C. $\{(1,\ 1,\ 0),\ (-1,\ 0,\ 1)\}$

D. $\{(-1,\ 0,\ 1)\}$

E. $\{\overset{e_1}{(1,\ 0,\ 0)},\ \overset{e_2}{(0,\ 1,\ 0)},\ \overset{e_3}{(0, 0,\ 1)}\}$

F. $\{(2,\ 2,\ 0),\ (-2,\ 0,\ 2)\}$

A is not correct since $(-1,0,1) \in U$ but $(-1,0,1) \notin \text{span}\{(1,1,0)\}$

B " since $\text{span}\{(1,1,0),(0,0,0)\} = \text{span}\{(1,1,0)\} \neq U$ (by A)

C ✓

D is not correct because $(1,1,0) \in U$ but $(1,1,0) \notin \text{span}\{(-1,0,1)\}$

E is not correct because $(1,0,0) \notin U$ but $(1,0,0) \in \text{span}\{e_1, e_2, e_3\}$.

F is also correct since $\text{span}\{(1,1,0), (-1,0,1)\} = \text{span}\{(2,2,0), (-2,0,2)\}$.

ANSWER $\boxed{C \text{ and } F}$

5. Which of the following statements are true?

I. The span of any two different vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$. FALSE

II. The set $\{(1,-2)\}$ spans a line through the origin in $\mathbf{R}^2$. TRUE

III. A set of vectors $\{u, v, w\}$ in a vector space spans V if every vector in V is a linear combination of u and $u+v+w$. TRUE

IV. The set $\left\{ \overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} \right\}$ spans $\mathbf{M}_{22}$. FALSE

V. The set $\{(1,1,1),(0,2,3)\}$ spans $\mathbf{R}^3$. FALSE : e.g. when we solve $(1,0,0) = a(1,1,1) + b(0,2,3)$ for a, b, we obtain equations

$$\left.\begin{aligned} a &= 1 \\ a+2b &= 0 \\ a+3b &= 0 \end{aligned}\right\} \Rightarrow b = 0, \text{ so } a = 0 \quad \text{This is impossible.}$$

and Hence $\{(1,1,1), (0,2,3)\}$ does not span $\mathbb{R}^3$.

ANSWER $\boxed{\text{II and III}}$

I is false: $(1,0)$ and $(0,0)$ are different but $\text{span}\{(1,0),(0,0)\} = \{(x,0) \mid x \in \mathbb{R}\}$ does not contain (e.g.) $(0,1)$.

II is true: $\text{span}\{(1,-2)\}$ is the line through $(0,0)$ with direction $(1,-2)$. (A cartesian eqn is $y=-2x$)

III is true: Let $v \in V$. If $v = au + b(u+v+w)$ for some $a, b \in \mathbb{R}$, then $v = (a+b)u + bv + bw \in \text{span}\{u,v,w\}$.

Thus $V \subseteq \text{span}\{u,v,w\} \subseteq V$ (the latter because $u,v,w \in V$)

Hence $V = \text{span}\{u,v,w\}$

IV is false: $\text{span}\{A,B\} = \left\{\begin{bmatrix} a & a \\ b & b \end{bmatrix} \middle| a,b \in \mathbb{R}\right\}$ which does not contain $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

V is false: see above

6. Let $u = (1, 0, 1)$, and $X = \{w \in \mathbf{R}^3 \mid w \times u = 0\}$

2 a) Is X a subspace of $\mathbf{R}^3$?

$2\frac{1}{2}$ b) If your answer to (a) is "yes", find a spanning set for X.

$1\frac{1}{2}$ c) Give a complete geometric description of X.

(You must justify your answers.)

a) Note that if $w = (x, y, z)$, then $w \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 0 & 1 \end{vmatrix} = (y, -(x-z), -y)$

$= (0,0,0) \Leftrightarrow y = 0$ and $x = z$, so $w = (x, 0, x)$ for some $x \in \mathbb{R}$.

Thus $W = \{(x,0,x) \mid x \in \mathbb{R}\} = \text{span}\{(1,0,1)\} = \text{span}\{u\}$; hence

W is a subspace. (OR: run the subspace test)

1/2 Justification $1\frac{1}{2}$

b) We saw in (a) that $\{u\}$ is a spanning set for W

(1) - any correct answer ($1\frac{1}{2}$) Justification

c) W is the line through 0 with direction $(1,0,1)$.

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

Note (1) part of the justification in (a) could simply be " $w \times u = 0 \Leftarrow w = ku$ for some $k \in \mathbb{R}$" (they should [may?] know this from high school)

(2) If they run the subspace test in (a), $1\frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$.

7. Consider the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$, with the standard operations. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Let $W = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-\pi) = f(\pi)\}$ be the subspace of functions which have the same value at $x = -\pi$ and $x = \pi$. Now define functions g, h, j and $k \in \mathbf{F}(\mathbf{R})$ by

$$g(x) = \sin x, \quad h(x) = \cos x,$$
$$k(x) = 1, \quad \text{and} \quad j(x) = \sin(x + \frac{\pi}{4}), \qquad \forall x \in \mathbf{R}.$$

1 a) Show that g and k belong to W.

2 b) Show that $j \in \operatorname{span}\{g, h\}$.

3 c) Show that $k \notin \operatorname{span}\{g, h\}$.

(You must justify your answers.)

1/2 1/2

a) Note that $g(-\pi) = 0 = g(\pi)$ and $k(-\pi) = 1 = k(\pi)$, so both g and k belong to W

b) Using the formula $\sin(x+a) = \cos a \sin x + \sin a \cos x$,

we see that $\sin(x + \pi/4) = \cos \pi/4 \sin x + \sin(\frac{\pi}{4}) \cos x$

1 - knowing what to do
1 - doing it properly

$= \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \quad (\forall x \in \mathbb{R})$

Hence $j = \frac{\sqrt{2}}{2} g + \frac{\sqrt{2}}{2} h \in \operatorname{span}\{g, h\}$

c) Suppose $a, b \in \mathbb{R}$ and $k = ag + bh$. Then

(1) - knowing what's needed

$k(x) = ag(x) + bh(x), \forall x \in \mathbb{R}$. In particular, (1) at

" $k = ag + bh$ " (*) (something like this)

$x = 0$, this implies $1 = a.0 + b.1$ and at

$x = \pi$ " $1 = a.0 + b.(-1)$, which

is impossible. Hence $k \notin \operatorname{span}\{g, h\}$

(1) - any set of eqns consistent with (*) that has no solution

8. [Bonus] Give the set $U = \{(x, x+2) \mid x \in \mathbf{R}\}$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x',\ y + y' - 2) \quad \text{(vector addition)}$$

and

$$k \odot (x,\ y) = (kx,\ ky - 2k + 2) \quad \text{(multiplication by scalars)}.$$

$[k \cdot (x, x+2) = (kx,\ k(x+2) - 2k + 2) = (kx,\ kx + 2) \in U]$

1 a) Prove that U is closed under the operation of addition defined above.

1 b) Show that there *is* a zero vector for U in U (i.e. Find it and show it works.).

2 c) Show that every element $(x, x+2) \in X$ has a negative in U.
(i.e. If $v = (x, x+2) \in U$, what is $-v \in U$?)

a) Suppose $u = (x, x+2)$ and $v = (x', x'+2)$ belong to U (for $x, x' \in \mathbb{R}$). Then $u \oplus v = (x+x', (x+2)+(x'+2)-2)$

$= (x+x', (x+x')+2) \in U$.

So U is closed under "$\oplus$".

1 (must be well written to obtain full marks)

b) Note that $\forall x \in \mathbb{R}$, $(x, x+2) \oplus (0, 2) = (x, x+2+2-2)$

$= (x, x+2)$.

Hence $(0,2)$ is the zero vector for U.

$\frac{1}{2}$ - correct zero (with justn) + $\frac{1}{2}$ justification

c) Note that $\forall x \in \mathbb{R}$, $(x, x+2) \oplus (-x, -x+4)$

$= (x - x,\ x + (-x+4) - 2)$

$= (0, 2)$, which is the zero vector for U. Hence $-(x, x+2) = (-x, -x+4)$.

$\frac{1}{2}$ - correct negative (with justn) + $1\frac{1}{2}$ well-written justn.