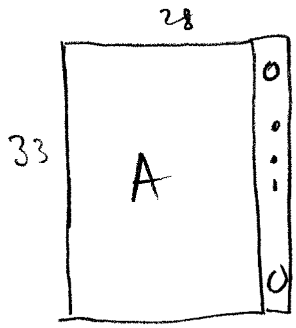
Mat 1341A Soln Ass#2 F13

1. If the coefficient matrix A in a homogeneous system of 33 equations in 28 unknowns is known to have rank 12, how many parameters are there in the general solution?



$$\#\text{ parameters in gen'l soln} = \#\text{ cols } A - \text{rank } A = 28 - 12 = 16$$

ANSWER $\boxed{16}$

2. For a *nonhomogeneous* system of 2013 equations in 3012 unknowns, answer the following three questions:

$[A \mid b]$, $b \neq 0$

- (I) Can the system be inconsistent? Yes – see below
- (II) Can the system have infinitely many solutions? Yes – see below
- Can the system have a unique solution? No – rank $A \leq 2013 < 3012$, so it's impossible to have rank A = rank $[A \mid b]$ = # cols A = 3012

A. Yes, Yes, No.
B. No, No, Yes.
C. Yes, No, Yes.
D. No, Yes, Yes.
E. Yes, Yes, Yes.
F. No, No, No.

(I) If $[A \mid b] = \left[\begin{array}{c|c} 0 & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$, the system is inconsistent.

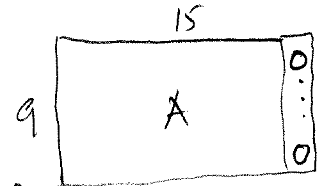
(II) If $[A \mid b] = \left[\begin{array}{ccc|c} 1 & \cdots & 0 & 1 \\ 0 & & \vdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right]$, the system has inf'ly many solns (with 3011 parameters).

ANSWER $\boxed{A}$

3. Let A be the 9×15 coefficient matrix of a *homogeneous* linear system, and suppose that this system has infinitely many solution with 8 parameters.

- What is the rank of A? (7 - see below)
- Do the columns of A, considered as vectors in $\mathbf{R}^9$, span $\mathbf{R}^9$?

(No - see below)



A. 0, Yes
B. 7, Yes
C. 7, No
D. 6, Yes
E. 6, No
F. 9, No

$$\#\text{parameters} = \#\text{cols} A - \text{rank} A, \text{ so}$$
$$\text{rank} A = \#\text{cols} A - \#\text{parameters} = 15 - 8 = 7$$

Do cols of A span $\mathbb{R}^9$?

This would mean that $[A|b]$ is consistent for every $b \in \mathbb{R}^9$. But $[A|b]$

$$\sim \begin{array}{c} 7\{ \\ 2\{ \end{array} \left[\begin{array}{ccc|c} 1 & \cdots & * & \tilde{b} \\ & & 1 & \\ 0 & \cdots & 0 & ** \end{array}\right]$$

, so it's possible that $*$ or $**$ will not be zero. Thus, no, the columns cannot span $\mathbb{R}^9$.

ANSWER $\boxed{C}$

4. Find a basis for the solution space of the equation $2x - 5y + 3z = 0$.

$$[2 \;\; -5 \;\; 3 \mid 0] \sim \begin{array}{c} \quad r \quad s \\ [1 \;\; -\tfrac{5}{2} \;\; -\tfrac{3}{2} \mid 0] \end{array}$$

$x = 5/2\, r + 3/2\, s$
$y = r$; $r, s \in \mathbb{R}$ $\therefore$ gen'l solution is
$z = s$

$$S = \{ (5/2\, r + 3/2\, s,\ r,\ s) \mid r, s \in \mathbb{R} \}$$
$$= \text{span}\{ (5/2, 1, 0), (3/2, 0, 1) \} = \text{span}\{ \underset{v_1}{(5,2,0)}, \underset{v_2}{(3,0,2)} \}$$

Moreover neither of the 2 vectors v_1, v_2 are multiples of the other so $\{v_1, v_2\}$ is l.i. Hence $\{v_1, v_2\}$ is a basis for S

ANSWER $\boxed{\{(5,2,0), (3,0,2)\}}$

5. Which of two the following three sets is a basis of $\mathbf{R}^3$?

$B_1 = \{(1,\ 0,\ 1),\ (6,\ 4,\ 5),\ (-4,\ -4,\ 7)\}$
$B_2 = \{(2,\ 1,\ 3),\ (3,\ 1,\ -3),\ (1,\ 1,\ 9)\}$
$B_3 = \{(3,\ -1,\ 2),\ (5,\ 1,\ 1),\ (1,\ 1,\ 1)\}$

Since each set has $\underbrace{3 = \dim \mathbb{R}^3}_{*}$ vectors, we need only check to see which sets are l.i., since by a theorem, they will then automatically (because of (*)) span $\mathbb{R}^3$.

$\underline{B_1}$: $a(1,0,1)+b(6,4,5)+c(-4,-4,7) = (0,0,0)$ is the linear system in variables a, b, c with augmented matrix $\left[\begin{array}{ccc|c} 1 & 6 & -4 & 0 \\ 0 & 4 & -4 & 0 \\ 1 & 5 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 6 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 11 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 6 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$, so $\operatorname{rank} A = \operatorname{rank}[A|b] = 3 = \#\text{ cols } A$; thus this system has a unique sol'n $(a,b,c) = (0,0,0)$. Thus B_1 is l.i., and hence is a basis

$\underline{B_2}$: We use the method we discovered in the sol'n for B_1: We solve

$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & -3 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This system has ∞ly many solution, so $a\underbrace{(2,1,3)}_{v_1}+b\underbrace{(3,1,-3)}_{v_2}+c\underbrace{(1,1,9)}_{v_3} = (0,0,0) \not\Rightarrow (a,b,c) = (0,0,0)$ (e.g. $-(2,1,3)+(3,1,-3)+(1,1,9) = (0,0,0)$). Hence B_2 is not l.i. and hence is not a basis for $\mathbb{R}^3$.

$\underline{B_3}$ Since we were told 2 of the sets were bases, B_3 must also be a basis.

[Directly; as above

$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \cdots \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$, so this system has the unique soln $(a,b,c) = (0,0,0)$. Hence B_3 is l.i. and is thus a basis of $\mathbb{R}^3$.

ANSWER $\boxed{B_1 \ \& \ B_3}$

6. Suppose $e, f \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & ez & = & f \\ x & & & + & z & = & -1 \\ 2x & + & y & + & z & = & -1 \end{array}$$

(You must justify all your answers.)

$2\frac{1}{3}$ a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for all values of e and f.

(✱)

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 3 & -2 & e & f \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & e-3 & f+3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & e-5 & f+5 \end{array}\right] \quad \frac{1}{2}$$

We can't proceed further without making assumptions about e and/or f.
So we consider the cases:

(I) If $e-5 \neq 0$, we can obtain another leading one in row 3 of A, so that rank $A = 3$ = rank $[A|b]$ in this case.

(II) If $e-5=0$, $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & f+5 \end{array}\right]$.

(i) If $f+5 \neq 0$,
rank $A = 2 < 3$ = rank $[A|b]$.

(ii) If $f+5=0$,
rank $A = 2$ = rank $[A|b]$.

In summary,

$$\text{rank } A = \begin{cases} 3 & \text{if } e \neq 5 \ (\forall f \in \mathbb{R}) \ \frac{1}{2} \\ 2 & \text{if } e = 5 \ (\forall f \in \mathbb{R}) \ \frac{1}{2} \end{cases}$$

and

$$\text{rank } [A|b] = \begin{cases} 3 & \text{if } e \neq 5 \text{ or } f \neq -5 \ \frac{1}{2} \\ 2 & \text{if } e = 5 \text{ and } f = -5 \ \frac{1}{2} \end{cases}$$

(As long as rank A, rank (A|b) are consistent with (✱) - if incorrect, please give the marks in the summary)

(Q.6 part (b) is on the next page...)

1½

6b). Using part (a), find all values of e and f so that this system has

(i) a unique solution, $\iff \operatorname{rank} A = \operatorname{rank}[A|b] = 3 = \#$ cols of A

$\iff e = 5$

½

(ii) infinitely many solutions, ~~or~~ $\iff \operatorname{rank} A = \operatorname{rank}[A|b] < \#$ cols of A

$\iff e = 5$ and $f = -5$

½

(iii) no solutions. $\iff \operatorname{rank} A < \operatorname{rank}[A\ b]$

$\iff e = 5$ and $f \neq -5$

½

1

(Answers in (b) need only be consistent with the answers in (a). However, if an error results in a considerable simplification, remove 2 pts.)

(Q.6 part (c) is on the next page.)

2

6c). In case b(ii) above, give a complete geometric description of the set of solutions.

In b(ii), $e=5$ and $f=-5$, so $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So the general solution is $\left.\begin{array}{l} x=-1+s \\ y=1+s \\ z=s \end{array}\right\}$, $s \in \mathbb{R}$ $\frac{1}{2}$

ie. $\{(-1+s, 1+s, s) \mid s \in \mathbb{R}\}$. This $\frac{1}{2}$

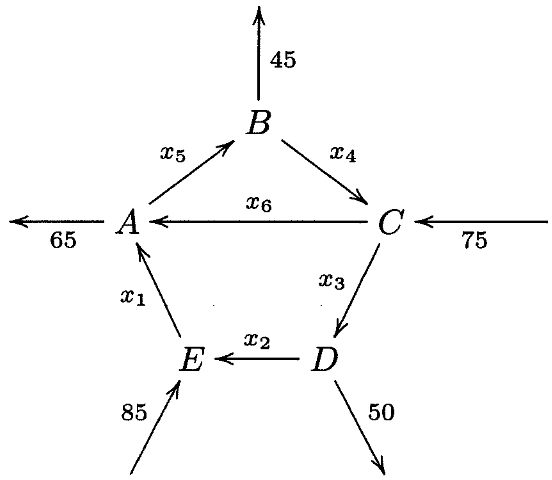
is the line in $\mathbb{R}^3$ through $(-1,1,0)$ with $\frac{1}{2}$

direction $(1,1,1)$.

$\frac{1}{2}$

(The last $1\frac{1}{2}$ marks can be given as long as the description is consistent with the general soln – even if the latter is incorrect, as long as the general solution given is not unique or all of $\mathbb{R}^3$.)

7. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(You must justify all your answers.)

$3\frac{1}{2}$ a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \dots, 6$.
(Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Intersection	Flow in	=	Flow out
A	$x_1 + x_6$	=	$65 + x_5$
B	x_5	=	$45 + x_4$
C	$x_4 + 75$	=	$x_3 + x_6$
D	x_3	=	$x_2 + 50$
E	$x_2 + 85$	=	x_1.

$5 @ \frac{1}{2} = 2\frac{1}{2}$

Since the streets are one-way, $X_i \geq 0$, $i = 1, \dots, 6$, and $\frac{1}{2}$

as the number of cars is an integer, $X_i \in \mathbb{Z}$, $i = 1, \dots, 6$). $\frac{1}{2}$

These can be summarized by
$X_i \in \mathbb{N} = \{0, 1, 2 \dots\}$, $i = 1, \dots, 6$

(Q.7 part (b) is on the next page.)

7(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

I

$$\begin{array}{cccccc|c} & & & & s & t & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 1 & 55 \\ 0 & 1 & 0 & 0 & -1 & 1 & -15 \\ 0 & 0 & 1 & 0 & -1 & 1 & 25 \\ 0 & 0 & 0 & 1 & -1 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

II (Correct RRE form!)

$$\begin{array}{cccccc|c} & & & & s & t & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 1 & 65 \\ 0 & 1 & 0 & 0 & -1 & 1 & -20 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 0 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

I

$$\begin{aligned} x_1 &= 55 + s - t \\ x_2 &= -15 + s - t \\ x_3 &= 25 + s - t \\ x_4 &= -35 + s \\ x_5 &= s \\ x_6 &= t \end{aligned} \quad ; \ s, t \in \mathbb{R}$$

6 @ 1/4 each
(allow one error
except for x_4.)

OR

II

$$\begin{aligned} x_1 &= 65 + s - t \\ x_2 &= -20 + s - t \\ x_3 &= 30 + s - t \\ x_4 &= -45 + s \\ x_5 &= s \\ x_6 &= t \end{aligned} \quad ; \ s, t \in \mathbb{R}$$

If the student knew the given RRE form was incorrect, and gives the correct RRE form, add 1 bonus point. (They can then give the soln to verify I or II.)

(Q.7 part (c) is on the next page.)

(1) **7(c).** If $\overline{ED}$ were closed due to roadwork, find the minimum flow along $\overline{AC}$ (x_6), using **your results from (b).**

$\overline{ED}$ closed $\Leftrightarrow x_2 = 0$.

(I) Here, $x_2 = -15 + s - t$, so $s - t = 15$. Thus $t = s - 15$ (*)

$x_1 = 70$
$x_2 = 0$
$x_3 = 40$
$x_4 = -35 + s$
$x_5 = s$
$x_6 = s - 15.$

We want to find the minimum flow along $\overline{AC}$, so we seek the smallest value of t, which occurs (by (*)) at the smallest value of s. Implementing the constraints $x_i \geq 0, i=1,..,6,$ the relevant constraints are

$x_4 \geq 0 \Leftrightarrow s \geq 35$
$x_5 \geq 0 \Leftrightarrow s \geq 0$
$x_6 \geq 0 \Leftrightarrow s \geq 15.$

Since all must be satisfied, $s \geq 35$

Hence, the smallest value for $x_6 = t = s - 15$ is $35 - 15 = \underline{20}$

(II) Here $x_2 = -20 + s - t$, so $s - t = 20$. Thus $t = s - 20$, (**)

and
$x_1 = 85$
$x_2 = 0$
$x_3 = 50$
$x_4 = -45 + s$
$x_5 = s$
$x_6 = s - 20.$

We seek the minimum value of $x_6 = t$, and hence (by (**)) the minimum value of s. Implementing the constraints $x_i \geq 0, i=1,\dots,6,$ the relevant constraints are

$x_4 \geq 0 \Leftrightarrow s \geq 45$
$x_5 \geq 0 \Leftrightarrow s \geq 0$
$x_6 \geq 0 \Leftrightarrow s \geq 20$

To satisfy all of these, $s \geq 45$.

Hence the smallest value for $x_6 = t = s - 20 = \underline{\underline{25}}.$

(1/2) - correct + (1/2) - justification

8. [Bonus] If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix such that the vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are linearly independent, prove carefully that rank $A = 2$. (You cannot choose the matrix A - your proof must work for every 2×2 matrix with the property above, i.e. every 2×2 matrix with independent columns.)

Since $\left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$ is l.i., $\begin{bmatrix} a \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We treat 2 cases: (I) $a \neq 0$ (II) $a = 0, c \neq 0$.

(I) $a \neq 0$ Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & b/a \\ 0 & d - \frac{bc}{a} \end{bmatrix}$. If $d = \frac{bc}{a}$, then $\begin{bmatrix} b \\ d \end{bmatrix} = \frac{b}{a}\begin{bmatrix} a \\ c \end{bmatrix}$, a contradiction to the independence of $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$. Hence $d \neq \frac{bc}{a}$, so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$, and thus rank $A = 2$.

(II) $a = 0$ and $c \neq 0$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \sim \begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix}$. If $b = 0$, then $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} = \frac{d}{c}\begin{bmatrix} 0 \\ c \end{bmatrix} = \frac{d}{c}\begin{bmatrix} a \\ c \end{bmatrix}$, a contradiction to the independence of $\begin{bmatrix} b \\ d \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix}$. Hence in this case $b \neq 0$ and so $A \sim \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$. Hence rank $A = 2$.

(1) Some correct idea + (1) Some progress + (1) all cases considered + (1) clearly presented