Multiple Choice Questions (1-4)

Question 1 *For what values of x is the following function continuous?*

$$f(x) = \begin{cases} x-2 & \text{if } x < -1 \\ x^2 - 2x + 1 & \text{if } -1 \le x \le 2 \\ 3 - x & \text{if } x > 2 \end{cases}$$

A) all real numbers **B)** all real numbers, except -1 **C)** all real numbers, except 2
D) all real numbers, except -1 and 2 **E)** all real numbers, except -1, 2 and 0.

For $x = -1$

$\lim_{x \mapsto -1^-} f(x) = -3 \neq \lim_{x \mapsto -1^+} f(x) = 4.$

So f is discontinuous @ $x = -1$

f is continuous everywhere else.

For $x = 2$

$\lim_{x \mapsto 2^-} f(x) = 1 = \lim_{x \mapsto 2^+} f(x)$

$f(2) = 1 = \lim_{x \mapsto 2} f(x)$

f is continuous @ $x = 2$.

Question 2 *Find the equation of the tangent line of the function $f(x) = x\sqrt{2x+5}$ at $x = 2$.*

A) $y = \frac{13}{3}x - \frac{8}{3}$ **B)** $y = \frac{11}{3}x - \frac{4}{3}$ **C)** $y = \frac{17}{3}x - \frac{2}{3}$ **D)** $y = \frac{13}{3}x + \frac{7}{3}$

E) $y = \frac{11}{3}x + \frac{2}{3}$

By product Rule $\rightarrow$ $f'(x) = \sqrt{2x+5} + x \cdot 2\left(\frac{1}{2\sqrt{2x+5}}\right)$

$f'(2) = 3 + \frac{2}{3} = \frac{11}{3}$

$f(2) = 2\sqrt{9} = 6$

$y - 6 = \frac{11}{3}(x - 2)$

$y = \frac{11}{3}x - \frac{22}{3} + 6$

2

$$\boxed{y = \frac{13}{3} - \frac{4}{3}}$$

Question 3 *Assuming that a bank is paying 5% annual interest which is compounded continuously, find the time needed for an initial deposit of 5,000 dollars to triple.*

A) $20\ln(2)$ years **B)** $10\ln(2)$ years **C)** $20\ln(3)$ years **D)** $10\ln(3)$ years

E) $16\ln(4)$ years

$$3P = Pe^{rt}$$

$$3 = e^{0.05t}$$

$$\Rightarrow \ln 3 = 0.05t$$

$$t = \frac{\ln 2}{5\times 10^{-2}} = 20\ln 3$$

Question 4 *Find the following limit.*

$$\lim_{x\to 4^-} \frac{|x-4|}{x-4}.$$

(Note: This is a one-sided limit)

A) 1 **B)** -1 **C)** 0 **D)** ∞ **E)** The limit does NOT exist.

$$|x-4| = \begin{cases} x-4 & x \geq 4 \\ -(x-4) & x < 4. \end{cases}$$

$$\lim_{x\mapsto 4^-} \frac{|x-4|}{x-4} = \lim_{x\mapsto 4} \frac{-(x-4)}{x-4} = -1$$

Long Answer Questions (5-7)

Question 5 (14 points)
Using only the definition of derivative as a limit, calculate $f'(x)$ where

$$f(x) = \sqrt{x-2}.$$

Solution

$$f'(x) = \lim_{\Delta x \mapsto 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =$$

$$\lim_{\Delta x \mapsto 0} \frac{\sqrt{x+\Delta x-2} - \sqrt{x-2}}{\Delta x} =$$

Rationalize:

$$= \lim_{\Delta x \mapsto 0} \frac{\sqrt{x+\Delta x-2} - \sqrt{x-2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x-2} + \sqrt{x-2}}{\sqrt{x+\Delta x-2} + \sqrt{x-2}}$$

$$= \lim_{\Delta x \mapsto 0} \frac{x+\Delta x-2 - (x-2)}{\Delta x\left(\sqrt{x+\Delta x-2} + \sqrt{x-2}\right)}$$

$$= \lim_{\Delta x \mapsto 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x-2} + \sqrt{x-2}\right)} = \frac{1}{2\sqrt{x-2}}$$

Question 6 **(a)(6 points)** *The cost of producing x units of a product is given by*

$$C(x) = \frac{4x}{2+\sqrt{x}},$$

where $C(x)$ is the cost in dollars. Find the marginal cost for producing 4 units.

$C'(4) = ?$

Solution

Q. R

$$C'(x) = \frac{4(2+\sqrt{x}) - 4x\left(\frac{1}{2\sqrt{x}}\right)}{(2+\sqrt{x})^2} = \frac{4(4) - 16\left(\frac{1}{4}\right)}{16}$$

$$= \frac{16-4}{16} = \frac{12}{16} = \boxed{\frac{3}{4}}$$

(b)(6 points) Solve the following logarithmic equation.

$$\ln(x) - \ln(x-1) = 1$$

Solution

$$\ln(x) - \ln(x-1) = \ln\frac{x}{x-1} = \ln \underbrace{e}^{1}$$

ln is 1-1 function

$$\frac{x}{x-1} = e \qquad \{x>1\}$$

$$x = e(x-1)$$

$$x - ex = -e$$

$$x(1-e) = -e$$

or $\begin{cases} x = \dfrac{-e}{1-e} \\ x = \dfrac{e}{e-1} \end{cases}$

Question 6 (12 points) *When the price of a brand of golf ball is 10 dollars per golf ball, 32,000 golf balls are sold. When the price is raised to 13 dollars, 26,000 golf balls are sold. A golf ball costs 4 dollars to make, and the owners of the golf ball company had an initial cost of 22,000 dollars.*

- *Find the demand function for this brand of golf ball. You may assume the demand is linear.*
- *Find the revenue and cost functions for this brand of golf ball.*
- *Find the profit function for this brand of golf ball.*

Solution

x = # of golf balls, p = price per golf ball

Demand is linear $p = D(x)$

x	p
32,000	10
26,000	13

$m = \frac{\Delta p}{\Delta x} = \frac{-3}{6{,}000} = \frac{-1}{2{,}000}$

$p = mx + b = \frac{-1}{2{,}000}x + b$

Plug in $(32{,}000, 10)$

$10 = \frac{-1}{2{,}000} \cdot 32{,}000 + b$

$b = 26$

So $p = D(x) = \frac{-1}{2{,}000}x + 26$

$R(x)$ = Revenue $= xD(x) = \frac{-1}{2{,}000}x^2 + 26x$

$C(x)$ = Cost $= 4x + 22{,}000$

$P(x)$ = Profit = Revenue − Cost $= R(x) - C(x) = \frac{-1}{2{,}000}x^2 + 26x - [4x + 22{,}000]$

So $P(x) = \frac{-1}{2{,}000}x^2 + 22x - 22{,}000$

Question 7 (14 points)

The function $y = f(x)$ is defined implicitly by

$$x^2 + 2y^2 = 3xy.$$

Find the equation of the tangent line to the graph determined by the above equation at $(1, 1)$.

Solution

$$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(3xy)$$

$$2x + 4y\frac{dy}{dx} = 3\left[y + x\frac{dy}{dx}\right]$$

$$\frac{dy}{dx}[4y - 3x] = 3y - 2x$$

$$\frac{dy}{dx} = \left.\frac{3y-2x}{4y-3x}\right|_{x=1} = \frac{1}{1} = 1.$$

$$y - 1 = 1(x-1) \Rightarrow$$

$$\boxed{y = x}$$

Multiple Choice Questions (1-4)

Question 1 *For what values of x is the following function continuous?*

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \le x \le 3 \\ 5 - x & \text{if } x > 3 \end{cases}$$

A) all real numbers **B)** all real numbers, except 1 **C)** all real numbers, except 3
D) all real numbers, except 1 and 3 **E)** all real numbers, except 1, 3 and 0.

For $x=1$

$\lim_{x \mapsto 1^-} f(x) = 2 = \lim_{x \mapsto 1^+} f(x)$

$f(1) = 2 = \lim_{x \mapsto 1} f(x)$

f is continuous @ $x=1$

For $X=3$

$\lim_{x \mapsto 3^-} f(x) = 4 \neq \lim_{x \mapsto 3^+} f(x) = 2$

f has discontinuity @ $x = 3$.

f is continuous everywhere else on $\mathbb{R}$.

Question 2 *Find the equation of the tangent line of the function $f(x) = x\sqrt{4x+1}$ at $x = 2$.*

A) $y = \frac{13}{3}x - \frac{8}{3}$ **B)** $y = \frac{17}{3}x + \frac{16}{3}$ **C)** $y = \frac{17}{3}x - \frac{2}{3}$ **D)** $y = \frac{13}{3}x + \frac{7}{3}$

E) $y = \frac{11}{3}x - \frac{2}{3}$

$$f'(x) = \sqrt{4x+1} + \left.\frac{x \cdot 4}{2\sqrt{4x+1}}\right|_{x=2} = 3 + \frac{4}{3} = \frac{13}{3}$$

$f'(2) = 2 \cdot \sqrt{9} = 6$

$y - 6 = \frac{13}{3}(x - 2)$

$y = \frac{13}{3}x - \frac{26}{3} + 6$

$y = \frac{13}{3}x - 8/3$

Question 3 *Assuming that a bank is paying 5% annual interest which is compounded continuously, find the time needed for an initial deposit of 3,000 dollars to double.*

A) $20\ln(2)$ years **B)** $10\ln(2)$ years **C)** $8\ln(3)$ years **D)** $12\ln(3)$ years

E) $16\ln(4)$ years

$$2P_0 = P_0 e^{rt}$$

$$2 = e^{0.05t}$$

$$\ln 2 = 0.05t \rightarrow t = \frac{\ln 2}{\underbrace{0.05}_{5\times 10^{-2}}} = 20\ln$$

Question 4 *Find the following limit.*

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2}.$$

(Note: This is a one-sided limit)

A) 1 B) -1 **C)** 0 **D)** ∞ **E)** The limit does NOT exist.

$$|x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases}$$

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2} \frac{-(x-2)}{x-2} = -1$$

Long Answer Questions (5-7)

Question 5 (14 points)
Using only the definition of derivative as a limit, calculate $f'(x)$ where

$$f(x) = \sqrt{x+1}.$$

Solution

$$f'(x) = \lim_{\Delta x \mapsto 0} \frac{\sqrt{x+1+\Delta x} - \sqrt{x+1}}{\Delta x}$$

Rationalize.

$$= \lim_{\Delta x \mapsto 0} \frac{\sqrt{x+1+\Delta x} - \sqrt{x+1}}{\Delta x} \cdot \frac{\sqrt{x+1+\Delta x} + \sqrt{x+1}}{\sqrt{x+1+\Delta x} + \sqrt{x+1}}$$

$$= \lim_{\Delta x \mapsto 0} \frac{(x+1+\Delta x) - (x+1)}{\Delta x\left(\sqrt{x+1+\Delta x} + \sqrt{x+1}\right)}$$

$$= \lim_{\Delta x \mapsto 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+1+\Delta x} + \sqrt{x+1}\right)}$$

$$= \boxed{\frac{+1}{2\sqrt{x+1}}}$$

Question 6 (a)(6 points) *The cost of producing x units of a product is given by*

$$C(x) = \frac{2x}{1+\sqrt{x}},$$

where $C(x)$ is the cost in dollars. Find the marginal cost for producing 4 units. $C'(4)$

Q. R

Solution

$$C'(x) = \frac{2(1+\sqrt{x}) - 2x\left(\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2} = \frac{2(3) - 8\left(\frac{1}{4}\right)}{9}$$

$$= 4/9$$

(b)(6 points) Solve the following logarithmic equation.

$$\ln(x+1) - \ln(x) = 1$$

Solution

$$\ln(x+1) - \ln x = 1 = \ln e$$

$$\ln \frac{x+1}{x} = \ln e.$$

ln is 1-1 function $\Rightarrow$

$$\frac{x+1}{x} = e$$

$$x+1 = e(x)$$

$$x - ex = -1$$

$$x(1-e) = -1$$

$$x = \frac{-1}{1-e} = \frac{1}{e-1}$$

either

Question 6 (12 points) *When the price of a brand of golf ball is 8 dollars per golf ball, 30,000 golf balls are sold. When the price is raised to 10 dollars, 26,000 golf balls are sold. A golf ball costs 4 dollars to make, and the owners of the golf ball company had an initial cost of 22,000 dollars.*

- *Find the demand function for this brand of golf ball. You may assume the demand is linear.*
- *Find the revenue and cost functions for this brand of golf ball.*
- *Find the profit function for this brand of golf ball.*

Solution Let x = # of golf balls
p = price / golf ball

$p = D(x)$ is linear

x	p
30,000	8
26,000	10

$$m = \frac{\Delta p}{\Delta x} = \frac{8-10}{30{,}000-26{,}000} = \frac{-1}{2{,}000}$$

$y = -\frac{1}{2{,}000}x + b$. Plug in $(30{,}000, 8)$

$$8 = \frac{-1}{2{,}000}(30{,}000) + b \Rightarrow b = 23$$

So $D(x) = \frac{-1}{2{,}000}x + 23$

Revenue $= R(x) = xD(x) = \frac{-1}{2{,}000}x^2 + 23x$

Cost $= C(x) = 22{,}000 + 4x$

Profit $= P(x) = R(x) - C(x) =$

$$\frac{-1}{2{,}000}x^2 + 23x - (22{,}000 + 4x)$$
$$= \frac{-1}{2{,}000}x^2 + 19x - 22{,}000$$

Question 7 (14 points)

The function $y = f(x)$ is defined implicitly by

$$x^2 + 3xy = 4y^2.$$

Find the equation of the tangent line to the graph determined by the above equation at $(1, 1)$.

Solution

$$\frac{d}{dx}(x^2 + 3xy) = \frac{d}{dx}(4y^2)$$

$$2x + 3\left(y + x\frac{dy}{dx}\right) = 8y\frac{dy}{dx}$$

$$2x + 3y = (-3x + 8y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \left.\frac{2x + 3y}{-3x + 8y}\right|_{x=1} = \frac{5}{5} = 1$$

$$y - 1 = 1(x - 1)$$

$$y = x$$