MAT 1330, Fall 2011 Assignment 6

Due November 23 at the beginning of class.

Late assignments will **not** be accepted; **nor** will unstapled assignments.

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Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. Consider the function $f(x) = x^3 - 5x^2 + 6x$ of a single real variable x.

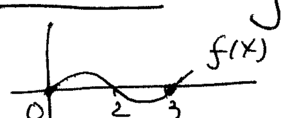
Show that there exist two distinct solutions of $f'(x) = 0$.

Method 1: $f'(x) = 3x^2 - 10x + 6$

$\Delta = 100 - 4 \cdot 3 \cdot 6 = 28 > 0$

$x_1 = \frac{10 + \sqrt{28}}{6}$, $x_2 = \frac{10 - \sqrt{28}}{6}$ – the two distict solutions of $f'(x) = 0$

Method 2: $f(x) = x^3 - 5x^2 + 6x = x(x-2)(x-3)$



$f(0) = f(2) = f(3) = 0$

$f(x)$ is continuous on $[0, 2]$ and $[2, 3]$ and it differentiable on $(0, 2)$ and $(2, 3)$, $f(0) = f(2)$ and $f(2) = f(3)$, then there exist $c_1 \in (0, 2)$ and $c_2 \in (2, 3)$ such that $f'(c_1) = 0$ and $f'(c_2) = 0$ (by Rolle's theorem).

QUESTION 2.

a) Find the Taylor polynomial of degree n of $f(x) = e^x$ near $x = 1$. $f(x)=e^x$

Taylor Polynomial of Degree n approximating $f(x)$ for x near a has the following form:

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

In our case, $a = 1$, $f(a) = f'(a) = \cdots f^{(n)}(a) = e$

$$P_n(x) = e + \frac{e(x-1)}{1!} + \frac{e(x-1)^2}{2!} + \cdots \frac{e(x-1)^n}{n!}$$

for x near 1,

$f(x) = e^x \approx P_n(x)$.

b) Approximate $\sqrt{e^3}$ using the Taylor polynomial obtained in (a) of degree $n = 3$.

For $n=3$

$$P_3(x) = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{3!} =$$

$$= e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6}$$

$$f\left(\tfrac{3}{2}\right) = e^{\frac{3}{2}} \approx P_3\left(\tfrac{3}{2}\right) = e + e\left(\tfrac{3}{2}-1\right) + \frac{e\left(\frac{3}{2}-1\right)^2}{2} + \frac{e\left(\frac{3}{2}-1\right)^3}{6} = 4.473839$$

QUESTION 3. Complete the following steps needed to sketch the graph of the function

$$f(x) = e^{-\frac{x^2-3x+1}{2}}. = e^{\frac{-x^2+3x-1}{2}}$$

a) Determine the domain of $f(x)$:

Answer: $\mathbb{R}$

b) Determine the vertical asymptotes of $f(x)$:

f(x) is defined everywhere.
There are no vertical asymptotes of f(x).

c) Determine the horizontal asymptote(s):

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-\frac{x^2}{2}\left(1-\frac{3}{x}+\frac{1}{x^2}\right)} = 0$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{-\frac{x^2}{2}\left(1-\frac{3}{x}+\frac{1}{x^2}\right)} = 0$

Answer: $y = 0$.

d) Determine the zeros of f (if any) and the value of f at $x = 0$:

Zeros of f: none. $f(0) = e^{-\frac{1}{2}}$. $= \frac{1}{\sqrt{e}}$

e) Compute the derivative $f'(x)$, find the critical points of $f(x)$, and determine the sign of $f'(x)$ and behavior of $f(x)$.

$f'(x) = e^{\frac{-x^2+3x-1}{2}} \cdot \left(\frac{3x-x^2-1}{2}\right)' = e^{\frac{-x^2+3x-1}{2}}\left(\frac{3-2x}{2}\right)$.

Critical point(s): $x = \frac{3}{2}$.

x	$-\infty$	$\frac{3}{2}$	$+\infty$
sign of $f'(x)$	+	0	−
behaviour of $f(x)$	↑	1.868	↓

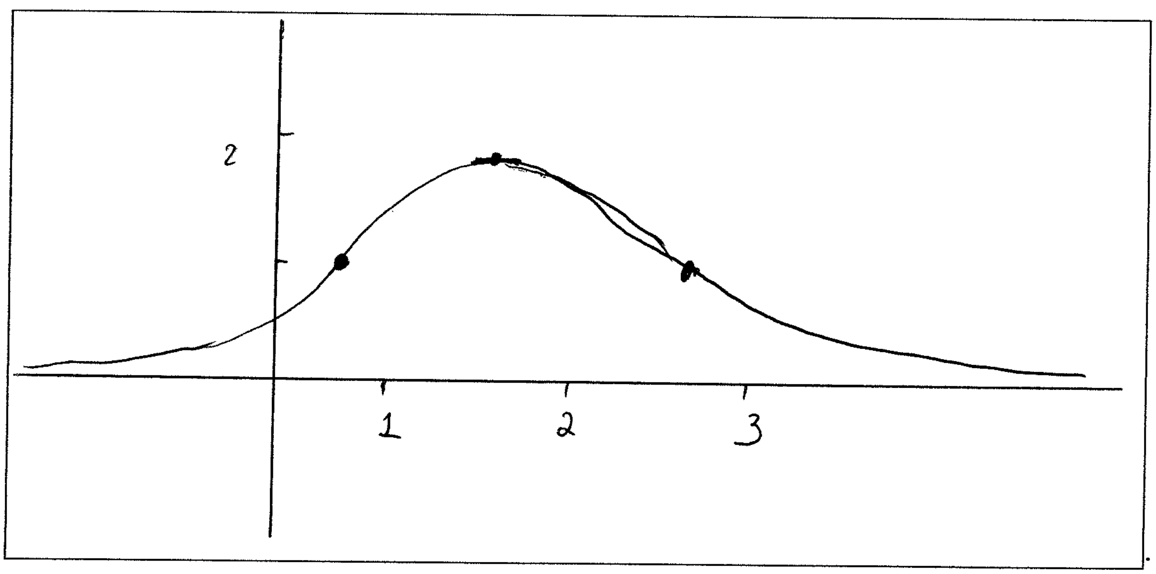
f) Compute the second derivative $f''(x)$, determine its sign, find the inflection points and determine the concavity of $f(x)$.

$$f''(x) = e^{\frac{-x^2+3x-1}{2}} \cdot \left(\frac{3-2x}{2}\right)^2 + e^{\frac{-x^2+3x-1}{2}}(-1) = e^{\frac{-x^2+3x-1}{2}}\left(\frac{(3-2x)^2}{2} - 1\right) = e^{\frac{-x^2+3x-1}{2}}(x-0.5)(x-2.5).$$

x	$-\infty$	0.5		2.5	$+\infty$
sign of $f''(x)$	+	0	−	0	+
concavity of $f(x)$	Concave up		Concave down		Concave up.

Inflection points: $x_1 = 0.5$ and $x_2 = 2.5$.

g) Sketch the graph of $f(x)$.



QUESTION 4. Determine the following limits: (a) $\lim_{x\to 0} \frac{1-\cos(x)}{x\tan(x)}$

$\lim_{x\to 0} \frac{1-\cos(x)}{x\tan(x)}$ This limit is of the form $\left[\frac{0}{0}\right]$. Applying l'Hospital's rule yields $\lim_{x\to 0} \frac{1-\cos(x)}{x\tan(x)} = \lim_{x\to 0} \frac{(1-\cos(x))'}{(x\tan(x))'} =$

$= \lim_{x\to 0} \frac{\sin x}{\tan x + \frac{x}{\cos^2 x}} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{\cos x}{\frac{1}{\cos^2 x} + \frac{1}{\cos^2 x} + 2\tan x \frac{x}{\cos^2 x}} =$

$= \frac{1}{2}$

(b) $\lim_{t\to 0} \frac{7^t - 6^t}{t}$

$\lim_{t\to 0} \frac{7^t - 6^t}{t} = \left[\frac{0}{0}\right] = \lim_{t\to 0} \frac{7^t \ln 7 - 6^t \ln 6}{1} =$

$= \ln 7 - \ln 6 = \ln\frac{7}{6} = 0.15415.$

(c) $\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos(2\pi x)}$

$\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos(2\pi x)} = \frac{1 - 1 + \ln 1}{1 + \cos 2\pi} = \frac{0}{2} = 0.$