MAT 1332 Summer 2011 - Practice Final

Question 1. Determine which of the following improper integrals converge.

(1) $\displaystyle\int_1^\infty \frac{dx}{x^{17}}$ (2) $\displaystyle\int_0^1 \frac{dx}{x^{17}}$ (3) $\displaystyle\int_5^\infty \frac{dx}{(2x-8)^{0.2}}$

(4) $\displaystyle\int_1^3 \frac{dx}{(x-1)^{\frac{1}{17}}}$ (5) $\displaystyle\int_{-\infty}^0 e^t dt$ (6) $\displaystyle\int_0^1 \frac{dx}{x}$

A: (1),(5),(6) B: (2),(3),(5) C: (2),(4),(6) D: (1),(4),(5) E: (3),(4),(5),(6)

Question 2. Find the value of the following integral:

$$\int_1^{\sqrt{e}} x\ln(x^2)dx.$$

A: 0 B: $e+\frac{1}{2}$ C: $\frac{1}{2}$ D: $\frac{e}{2}-\frac{1}{2}$ E: $\frac{e}{2}+\frac{1}{2}$

Question 3. Find the value of the integral

$$I = \int \frac{6x^2-26x+18}{3x^2-13x+4}dx.$$

Question 4. Find the real part of

$$z = \frac{6-17i}{(1-2i)^2}.$$

A: 2 B: 3 C: -3 D: -2 E: 0

Question 5. Find the area between the curves $y=(x-1)^2$ and $y=5-x^2$ (hint: find the points of intersection first).

Question 6. Find the volume of the solid obtained by rotating the region bounded by the given curve about the x-axis, and sketch the region:

$$y=4-x^2,\quad y=0,\quad x=0 \quad \text{(in the first quadrant)}.$$

Question 7. Suppose that a fish population grows according to the logistic equation, and that a fraction of the fish population is removed per unit time. Then the number of fish is given by the differential equation

$$\frac{dN}{dt} = 2N\left(1 - \frac{N}{10}\right) - HN.$$

(a) Find all biologically meaningful equilibria and their stability for $H = 1$.
(b) Draw the phase-line diagram for $H = 1$. Draw the solution curve with $N(0) = 1$ as a function of time.

Question 8. Consider the following matrices:

$$X = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 7 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 7 & 7 & 1 \end{bmatrix}$$

and the following statements:

(i) XY is defined;
(ii) XZ is defined;
(iii) $ZX^\top$ is defined;
(iv) Y is invertible;
(v) $Z + XX^\top$ is defined.

Which if these statement are correct?

A: (i),(iii),(v) B: (i),(iv) C: (ii) D: (ii),(iii) E: (iii),(v)

Question 9. Solve $\dfrac{dy}{dx} = 2 - 3y$, where $x_0 = 1$, $y_0 = 1$.

Question 10. Find the largest possible domain and the corresponding range of the function

$$z = f(x, y) = \sqrt{x} + \sqrt{y}.$$

Question 11. Find the largest possible domain and the corresponding range of the function

$$z = f(x, y) = x^2 + 2x + y^2 = (x + 1)^2 + y^2 - 1.$$

Describe the level curves $f(x, y) = C$, sketch a few of them, describe the possible values of C.

Question 12. Find the tangent plane to the graph of the function $f(x, y) = 3\pi - 3\cos 2x + 2\sin 3y$ at the point $(0, \pi)$.

A: $z = -\pi + 6x$
B: $z = -3 + 9\pi - 6y$
C: $z = 3 - 9\pi - 6y$
D: $z = 9\pi + 6x$
E: $z = -3\pi - 9y$
F: $z = 3 + 9\pi + 6x - 6y$.

Question 13. Find the $(2, 2)$-entry of the Jacobi matrix of the function

$$F(x, y) = \begin{bmatrix} x^2y + 2xe^y \\ \frac{x}{y} - 3ye^{-x} \end{bmatrix}$$

at the point $(2, 1)$.

A: $4 + 4e$
B: $-2 - 3e^{-2}$
C: 0
D: $4 + 2e$
E: $1 + 3e^{-2}$
F: $4 + 2e^{-1}$

Question 14. Consider the following system of linear differential equations:

$$\begin{aligned} \frac{dx}{dt} &= -x + 4y \\ \frac{dy}{dt} &= x - y. \end{aligned}$$

(a) Find the eigenvalues and eigenvectors associated with the system.

(b) Write down the general solution formula for the system.

(c) Give the particular solution for the initial values $x(0) = 7$, $y(0) = -2$.

(d) Draw the x- and y-nullclines and the direction arrows in the phase plane.

(e) Sketch the solution curve for the initial condition in part (c) in the phase plane.

(f) Is the point $(0, 0)$ stable or unstable. What is it called?

Question 15. Consider a disease that propagates according to the system

$$\begin{aligned}\frac{dx}{dt} &= 20 - 5xy - 5x \\ \frac{dy}{dt} &= 5xy - 10y,\end{aligned}$$

where x represents susceptible individuals and y represents infected individuals.

(a) Find the two biologically meaningful steady states;

(b) Find the Jacobi matrix of the system;

(c) For each of the two steady states from (a) find the eigenvalues of the Jacobi matrix;

(d) Determine the stability of the two steady states.