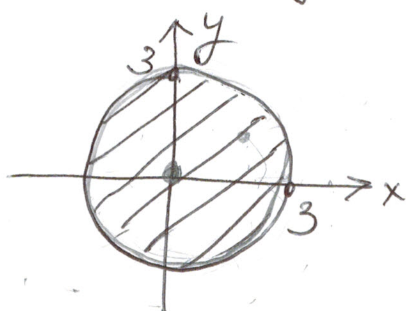
HW 5

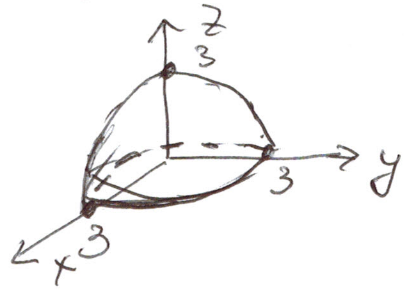
Q 1 $z = f(x,y) = \sqrt{9-x^2-y^2}$ – top of a sphere with radius $r=3$

Domain: $9 - x^2 - y^2 \geq 0$

$x^2+y^2 \leq 9$ – closed disk in the x-y-plane

$D = \{ (x,y) \in \mathbb{R}^2 : \; x^2+y^2 \leq 9 \}$





Range: $z \in [0,3]$

(when $x^2+y^2=9$ then $z=0$,

when $(x,y)=(0,0)$ $z=3$

$x^2+y^2<9$ then $z \in (0,3)$)

Level sets: $z = f(x,y) = c$

$\sqrt{9-x^2-y^2} = c$

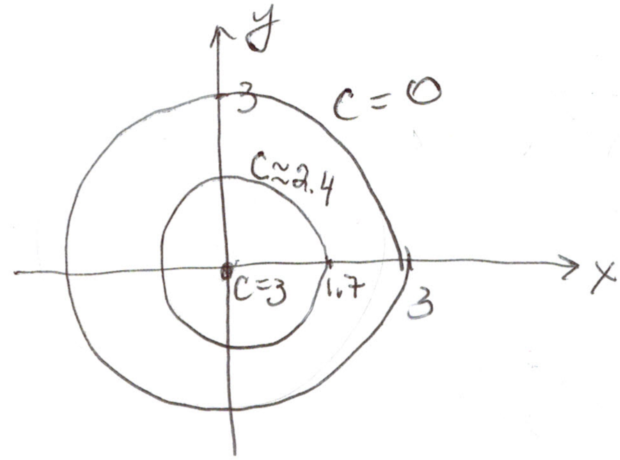
$9 - x^2 - y^2 = c^2$

$x^2+y^2 = 9 - c^2$ – these are circles with radius $\sqrt{9-c^2}$.

$c = 0$ $\quad r = \sqrt{9-c^2} = 3$

$c = 3$ $\quad r = 0$

$c = \sqrt{6} \approx 2.4$ $\quad r = \sqrt{3} \approx 1.7$



$\boxed{Q2}$ $z = f(x,y) = \tan(2x - 3y^2) + 2\pi$

$(x_0, y_0) = (0,0)$

Find the tangent plane to the graph of the function at (x_0, y_0)

$$\boxed{z = z_0 + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)}$$

$$z_0 = f(x_0,y_0) = \left[\tan(2x - 3y^2) + 2\pi\right]\Big|_{(0,0)} =$$

$$= \tan(2\cdot 0 - 3\cdot 0) + 2\pi = \tan 0 + 2\pi = 2\pi.$$

$$f'_x\Big|_{(0,0)} = \frac{1}{\cos^2(2x - 3y^2)} \cdot 2 \Big|_{(0,0)} = \frac{2}{\cos^2(0)} = \frac{2}{1} = 2$$

$$f'_y\Big|_{(0,0)} = \frac{1}{\cos^2(2x - 3y^2)} \cdot (-6y)\Big|_{(0,0)} = \frac{0}{1} = 0$$

Thus, the equation of the tangent plane is

$$z = 2\pi + 2x = 2(\pi + x).$$

$\boxed{Q3}$

$$F(x,y) = \begin{bmatrix} x^2 e^y + 2x\sin(x^y) \\ \sin(x^2) - 3ye^{-x} \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$$

Find the Jacobian matrix

$$J_{(x,y)} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}, \text{ where}$$

$$f_x = 2xe^y + 2\sin(x^y) + 2x\cos(x^y)\cdot(x^y)'_x =$$
$$= 2xe^y + 2\sin(x^y) + 2x\cos(x^y)\cdot y\cdot x^{y-1} =$$
$$= 2xe^y + 2\sin(x^y) + 2y\cos(x^y)\cdot x^y$$

$$f_y = x^2e^y + 2x\cdot\cos(x^y)\cdot(x^y)'_y =$$
$$= x^2e^y + 2x\cdot\cos(x^y)\cdot x^y\ln x$$

$$g_x = \cos(x^2)\cdot 2x + 3ye^{-x}$$
$$g_y = -3e^{-x}$$

$$\begin{cases} \dfrac{dx}{dt} = -3x+y \\ \dfrac{dy}{dt} = 4x-3y \end{cases}$$

(a) Find the eigenvalues and eigenvectors associated with the system.

$$A = \begin{bmatrix} -3 & 1 \\ 4 & -3 \end{bmatrix}$$

$$(A - \lambda I)\bar{v} = \bar{0} \quad , \quad \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -3-\lambda & 1 \\ 4 & -3-\lambda \end{bmatrix} = (-3-\lambda)(-3-\lambda) - 4 = 0$$

$$(3+\lambda)^2 - 4 = \lambda^2 + 6\lambda + 5 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -5$$

$\boxed{\lambda_1 = -1}$ $\quad (A - \lambda_1 I)\bar{v}^1 = \bar{0}$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

RRE form

$$\bar{v}^1 = \begin{bmatrix} \frac{s}{2} \\ s \end{bmatrix}, \ s \in \mathbb{R}, \ s \neq 0$$

$v_2^1 = s$

$v_1^1 = \frac{1}{2} v_2^1 = \frac{s}{2}$

Take $s=2$, $\bar{v}^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\lambda_2 = -5}$ $\quad (A - \lambda_2 I)\bar{v}^2 = \bar{0}$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

RRE form

$v_2^2 = S$ $\qquad 2v_1^2 + v_2^2 = 0$

$$2v_1^2 = -v_2^2$$

$$v_1^2 = -\frac{v_2^2}{2} = -\frac{S}{2}$$

$$\bar{v}^2 = \begin{bmatrix} -\frac{S}{2} \\ S \end{bmatrix}, \quad S \in \mathbb{R}, \ S \neq 0$$

Take $\bar{v}^2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $(S = 2)$

(b) The general solution has the following form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} - c_2 e^{-5t} \\ 2c_1 e^{-t} + 2c_2 e^{-5t} \end{bmatrix}$$

(c) Give the particular solution for the initial values $\begin{bmatrix} x(0) = 4 \\ y(0) = 4 \end{bmatrix}$

For $t=0$, we have.

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ 2c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$(2 \times \underline{2})(\underline{2} \times 1)$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}, \quad \det(B) = 2 - 2(-1) = 4$$

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$B B^{-1} \stackrel{?}{=} I$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Yes, indeed.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$c_1 = 3$

$c_2 = -1$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 3e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(d) Draw the x- and y-nullclines and the direction field in the xy plane

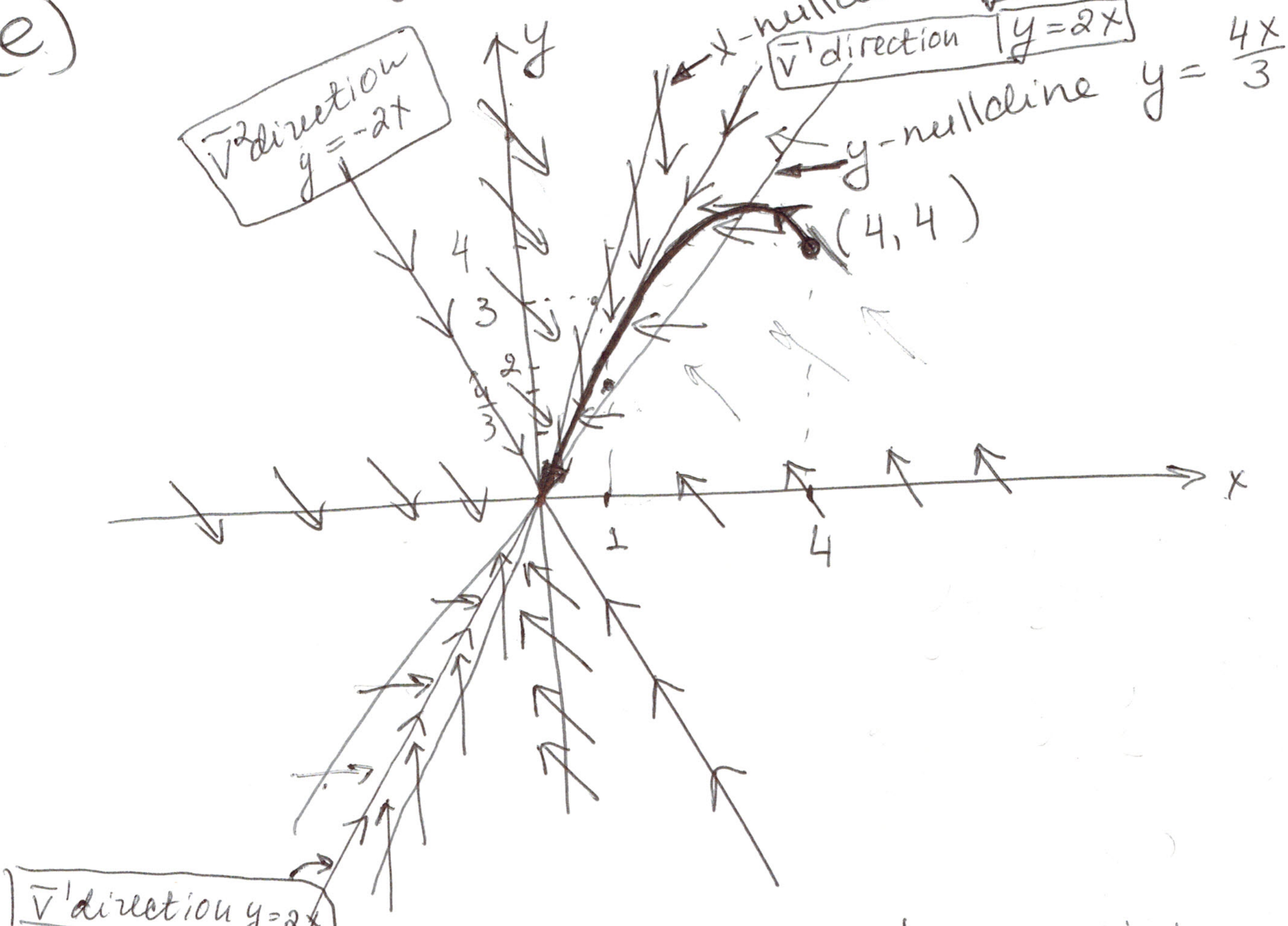
$\frac{dx}{dt} = 0 \qquad -3x + y = 0$

$y = 3x$ is the x-nullcline

$\frac{dy}{dt} = 0 \qquad 4x - 3y = 0$

$y = \frac{4}{3}x$ is the y-nullcline

(e)



(f) (0,0) is a stable node or sink

$$\frac{dy}{dt} = 0 \qquad y = \frac{4x}{3} \qquad \frac{dx}{dt} = -3x + y = -3x + \frac{4x}{3} = -\frac{5x}{3}$$

$$\frac{dx}{dt} < 0 \text{ if } x > 0$$

$$\frac{dx}{dt} > 0 \text{ if } x < 0$$

y-nullcline

$$x = 0 \qquad \frac{dx}{dt} = y. \qquad \frac{dy}{dt} = -3y$$

$\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$ if $y > 0$

$x \uparrow$ and $y \downarrow$. } $\boxed{\searrow \text{ if } y > 0.}$

$\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$ if $y < 0$

$x \downarrow$ and $y \uparrow$ } $\boxed{\nwarrow \text{ if } y < 0}$

$$y = 0 \qquad \frac{dx}{dt} = -3x \qquad \frac{dy}{dt} = 4x$$

$\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$ if $x > 0$

$x \downarrow$ and $y \uparrow$ $\boxed{\nwarrow \text{ if } x > 0}$

$\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$ if $x < 0$

$x \uparrow$ and $y \downarrow$ $\boxed{\searrow \text{ if } x < 0.}$

$$y = x \qquad \left\{ \begin{array}{l} \frac{dx}{dt} = -2x \\ \frac{dy}{dt} = x \end{array} \right. \qquad \text{if } x > 0 \quad \underbrace{x \downarrow \text{ and } y \uparrow}_{\nwarrow}$$

(5) $$\frac{dx}{dt} = 16 - 0.2xy - 0.4x = f(x,y)$$
$$\frac{dy}{dt} = 0.1xy - 8y = g(x,y)$$

(a) Find all biologically meaningful steady states:

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

$$\begin{cases} 16 - 0.2xy - 0.4x = 0 \\ 0.1xy - 8y = 0 \end{cases} \Rightarrow y(0.1x - 8) = 0$$
$$\hat{y}_1 = 0$$

- if $\hat{y}_1 = 0$ then $g(x, \hat{y}_1) = 0$
$$f(x, \hat{y}_1) = 16 - 0.4x = 0 \text{ if}$$
$$\hat{x}_1 = 40$$

Thus, the first equilibrium is
$$\boxed{(\hat{x}_1, \hat{y}_1) = (40, 0)}$$

- if ~~y≠0~~ $g(x,y) = y(0.1x - 8) = 0$ and $y \neq 0$
then $0.1x - 8 = 0$
$$\hat{x}_2 = 80$$

$f(\hat{x}_2, y) = 16 - 16y - 32 = 0$ if $\hat{y}_2 = -1$
Thus, the second equilibrium is $\boxed{(80, -1) = (\hat{x}_2, \hat{y}_2)}$

The second equilibrium is not biologically meaningful, since $\hat{y}_2 = -1$ is a negative number.

(b) Find the Jacobian matrix of the system:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -0.2y-0.4 & -0.2x \\ 0.1y & 0.1x-8. \end{bmatrix}$$

$$J(40,0) = \begin{bmatrix} -0.4 & -8 \\ 0 & -4 \end{bmatrix}$$

← this is an upper triangular matrix and we can find the eigenvalues using the rule.

Thus, $\lambda_1 = -0.4$ and $\lambda_2 = -4$.

(d) $(\hat{x}_1, \hat{y}_1) = (40, 1)$ is stable, it is a stable node or sink.