$\boxed{\text{Total : 17 points}}$

MAT 1332, Spring/Summer 2012 Assignment 3
Due June 12, 2012 at the beginning of class.
Late assignments will **not** be accepted; **nor** will unstapled assignments.

Instructor: Olga Vasilyeva

Student Name ______________________ Student Number ______________________

QUESTION 1. Consider the differential equation

$$\frac{dy}{dt} = y^2 + 9.$$

(a) Is this a pure time or an autonomous differential equation? Justify your answer in one sentence.

0.5

This is an autonomous dif. equation, because the RHS is a function of state variable only.

(b) Find the solution to the equation satisfying the initial condition $y(0) = 3$.

1.5

$$\frac{dy}{y^2+9} = dt$$

$$\int \frac{dy}{y^2+9} = \int dt$$

$$\int \frac{1\,dy}{9\cdot\left(\frac{y^2}{9}+1\right)} = \frac{1}{9}\int \frac{dy}{\left(\frac{y}{3}\right)^2+1} = \int dt$$

$$\boxed{\frac{y}{3} = z(y) \Rightarrow dy = 3dz}$$

$$\frac{1}{9}\int \frac{3dz}{z^2+1} = t + C$$

$$\frac{1}{3}\int \frac{dz}{z^2+1} = t + C$$

$$\frac{1}{3}\arctan(z) = t + C$$

$$\arctan\left(\frac{y}{3}\right) = 3t + 3C$$

$$\frac{y}{3} = \tan(3t+3C)$$

1

$y(0) = 3$

$3 = 3\tan(3C)$

$1 = \tan(3C)$ or

$3C = \arctan(1) = \frac{\pi}{4}$

$\boxed{C = \frac{\pi}{12}}$ ← found the consta[nt]

3

$y(t) = 3\tan(3t+3C)$ ← the general solution

$y(t) = 3\tan\left(3t + 3\cdot\frac{\pi}{12}\right) = 3\tan\left(3t+\frac{\pi}{4}\right)$ – the particula[r] solution.

QUESTION 2. Consider the differential

$$\frac{dy}{dx} = (y-1)e^x.$$

(a) Find the general solution to the equation.

$$\int \frac{dy}{y-1} = \int e^x dx$$

$$\ln|y-1| = e^x + C$$

$$e^{\ln|y-1|} = e^{e^x + C} = e^{e^x} \cdot e^C = e^{e^x} \cdot \bar{C}$$

$$|y-1| = \bar{C} \cdot e^{e^x}$$

$$y-1 = \pm \bar{C} \cdot e^{e^x} = \hat{C} \cdot e^{e^x}$$

$$\boxed{y(x) = \hat{C} \cdot e^{e^x} + 1}$$ ← the general solution

1

(b) Find the particular solution satisfying the initial condition $y(0) = e$.

$y(0) = e$.

$$\hat{C} \cdot e^{e^0} + 1 = \hat{C} \cdot e^1 + 1 = \hat{C} \cdot e + 1 = e$$

$$\hat{C} e = e - 1$$

$$\hat{C} = \frac{e-1}{e} = \left(1 - \frac{1}{e}\right)$$

Thus, $y(x) = \left(1 - \frac{1}{e}\right) \cdot e^{e^x} + 1$ – the particular solutions satisfying the initial condition.

1

2

QUESTION 3. Consider the differential

$$\frac{dy}{dt} = \cos t \cdot y^2.$$

(a) Find the general solution to the equation.

$$\frac{dy}{y^2} = \cos t\, dt$$

$$\int \frac{dy}{y^2} = \int \cos t\, dt$$

$$-\frac{1}{y} = \sin t + C.$$

$$\frac{1}{y} = -\sin t - C$$

$$y = -\frac{1}{\sin t + C}$$ ← the general solution

1

(b) Find the particular solution satisfying the initial condition $y(\frac{\pi}{2}) = 1$.

$$y(t) = -\frac{1}{\sin t + C}$$

$$y\left(\frac{\pi}{2}\right) = -\frac{1}{\sin(\frac{\pi}{2}) + C} = 1.$$

$$-\frac{1}{1+C} = 1 \quad \text{or} \quad \frac{1}{1+C} = -1$$

1

$$-(1+C) = +1.$$

$$1 + C = -1$$

$$C = -2$$

$$\boxed{y(t) = -\frac{1}{\sin t - 2} = \frac{1}{2 - \sin t}.}$$ ← the particular solution

2

QUESTION 4. Consider the differential equation

$$\frac{dy}{dx} = y(y-1) = f(y).$$

(a) Find all equilibria.

$f(y) = 0$
$y(y-1) = 0$
$y_1^* = 0 \;;\; y_2^* = 1$

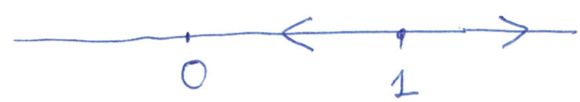
0,5

(b) Determine the stability of each equilibrium (you may use the stability criterion or the test point method).

$f'(y) = (y-1) + y = 2y - 1$
$f'(y_1^*) = f'(0) = -1 < 0 \Rightarrow y_1^*$ is stable
$f'(y_2^*) = f'(1) = 1 > 0 \Rightarrow y_2^*$ is unstable

$\boxed{\text{or}}$

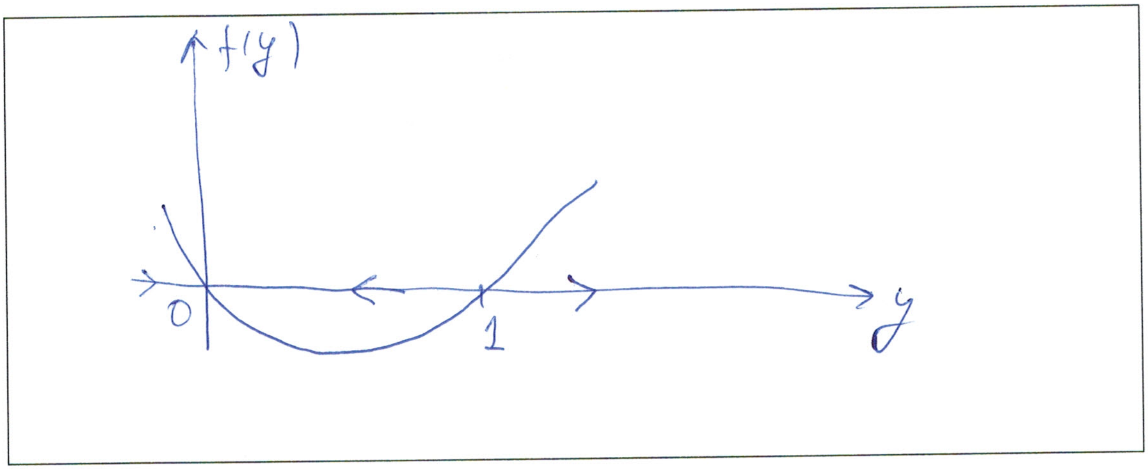
1,5



take $\bar{y} \in (0,1)$ $\bar{y} = 0.5$ find $f(0.5) = -0.25 < 0 \Rightarrow$
$\frac{dy}{dx} < 0$ for $y \in (0,1)$ and $y(x)$ is decreasing ($y \leftarrow$)
for all $y \in (0,1)$

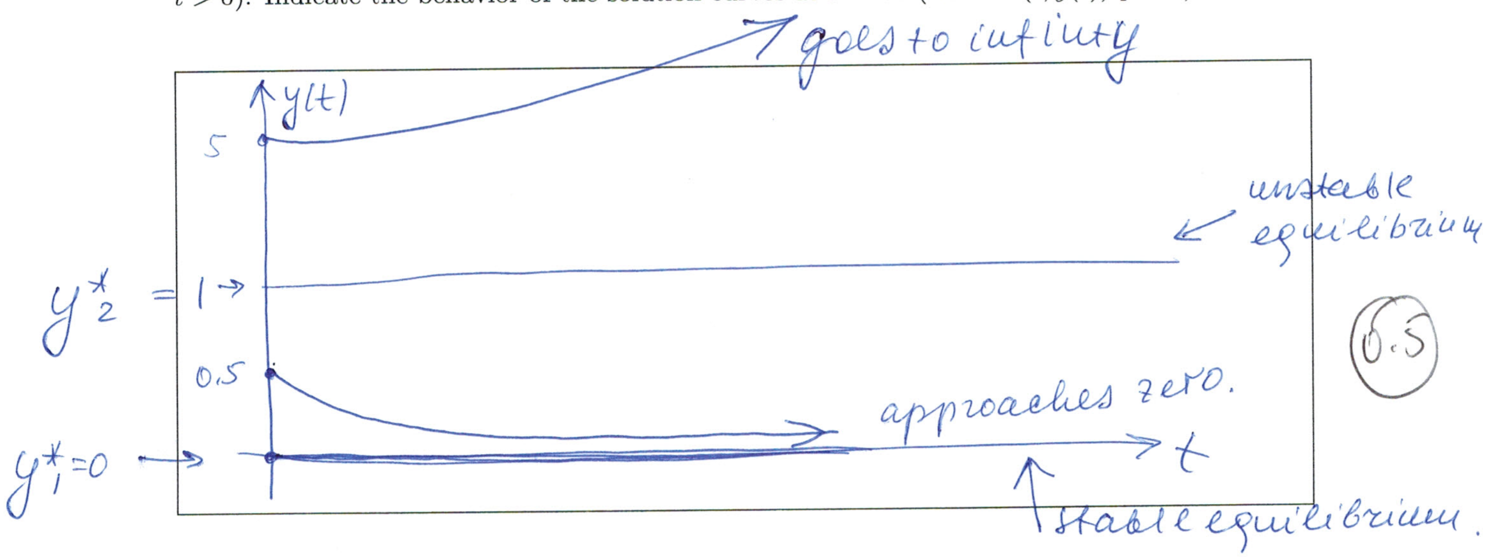
take $\bar{y} \in (1,+\infty)$ $\bar{y} = 2$ find $f(2) > 0 \Rightarrow$
$\frac{dy}{dx} > 0$ for all $y \in (1,+\infty)$ $y(x) \longrightarrow$ on this
infinite interval

(c) Draw the phase-live diagram for the differential equation in the $(y, f(y))$-plane.



0.5

(d) Graph the equilibrium solutions and solutions starting at $y(0) = \cancel{1}$ 0.5 and $y(0) = 5$ (for $t > 0$). Indicate the behavior of the solution curves as $t \to \infty$ (use the $(t, y(t))$-plane).



0.5

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QUESTION 5. Suppose the population of certain organisms follows the model

$$\frac{dN}{dt} = \frac{10N^2}{N^2+1} - 3N, = f(N)$$

where N is measured in thousands.

(a) Find the equilibria.

1

$$f(N) = 0 \qquad \frac{10N^2}{N^2+1} - 3N = 0$$

$$N\left(\frac{10N}{N^2+1} - 3\right) = 0$$

$$N_1^* = 0 \ ; \quad \frac{10N}{N^2+1} - 3 = 0 \Rightarrow 10N - 3N^2 - 3 = 0$$

$$3N^2 - 10N + 3 = 0$$

$$D = 100 - 4 \cdot 3 \cdot 3 = 64 > 0$$

(b) ~~Draw the phase-line diagram for the following differential~~ equation.

$$N_2^* = \frac{10+8}{6} = 3 \ ; \quad N_3^* = \frac{10-8}{6} = \frac{1}{3}$$

1,5

$$f'(N) = \left(\frac{10N^2}{N^2+1}\right)' - 3 = \frac{20N \cdot (N^2+1) - 10N^2 \cdot 2N}{(N^2+1)^2} - 3 =$$

$$= \frac{20N}{(N^2+1)^2} - 3$$

$$f'(0) = -3 \Rightarrow N_1^* = 0 \text{ is stable}$$

$$f'(3) = \frac{20 \cdot 3}{100} - 3 < 0 \Rightarrow N_2^* = 3 \text{ is stable}$$

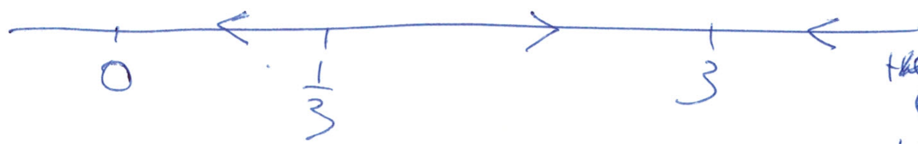
$$f'\left(\frac{1}{3}\right) = \frac{20 \cdot \frac{1}{3}}{\left(\left(\frac{1}{3}\right)^2 + 1\right)^2} - 3 = \frac{27}{5} - 3 > 0 \Rightarrow$$

$N_3^* = \frac{1}{3}$ is unstable equilibrium.

Draw the phase-line diagram for the dif. eq.

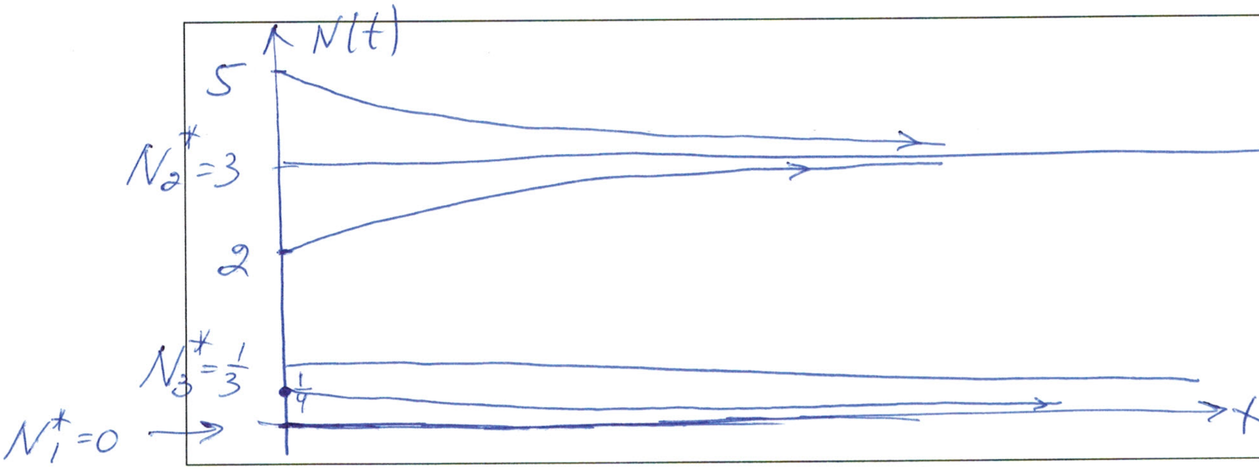
(c) ~~Determine the stability of equilibria using the stability criterion.~~

0.5



the N axis.

(d) Graph the equilibria and solutions starting at $N(0) = \frac{1}{4}$, $N(0) = 2$ and $N(0) = 5$ (use the $(t, N(t))$-plane.



1

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Question 6. A yam is put in a 200°C oven and heats up according to the differential equation

$$\frac{dH}{dt} = \alpha(200 - H).$$

(a) If the yam is at 20°C when it is put in the oven, solve the differential equation. (Your answer should include the constant α.)

$H_0 = H(0) = 20^\circ C.$

①

$\frac{dH}{dt} = \alpha(\underbrace{200-H}_{>0}) > 0. \Rightarrow \boxed{\alpha > 0}$

$\frac{dH}{200-H} = \alpha dt$

$\int \frac{dH}{200-H} = -\ln|200-H| = \alpha t + c$

$\ln|200-H| = -\alpha t - c$

$|200-H| = e^{-\alpha t - c} = e^{-\alpha t} \cdot e^{-c} = \bar{c} \cdot e^{-\alpha t}$

$200 - H = \pm \bar{c} e^{-\alpha t} = \hat{c} \cdot e^{-\alpha t}$

$H(t) = 200 - \hat{c} \cdot e^{-\alpha t}$

$H(0) = 20^\circ C$

0.5

$20 = 200 - \hat{c} \cdot e^{-\alpha \cdot 0} = 200 - \hat{c} \Rightarrow$

$\boxed{\hat{c} = 180}$

$\boxed{H(t) = 200 - 180 e^{-\alpha t}}$

(b) Find α using the fact that after 30 min the temperature of the yam is 120°C.

0.5

$$H(30) = 120°C \qquad H(t) = 200 - 180e^{-0.027t}$$

$$200 - 180e^{-30\alpha} = 120$$

$$80 = 180e^{-30\alpha}$$

$$e^{-30\alpha} = \frac{80}{180} = \frac{4}{9}$$

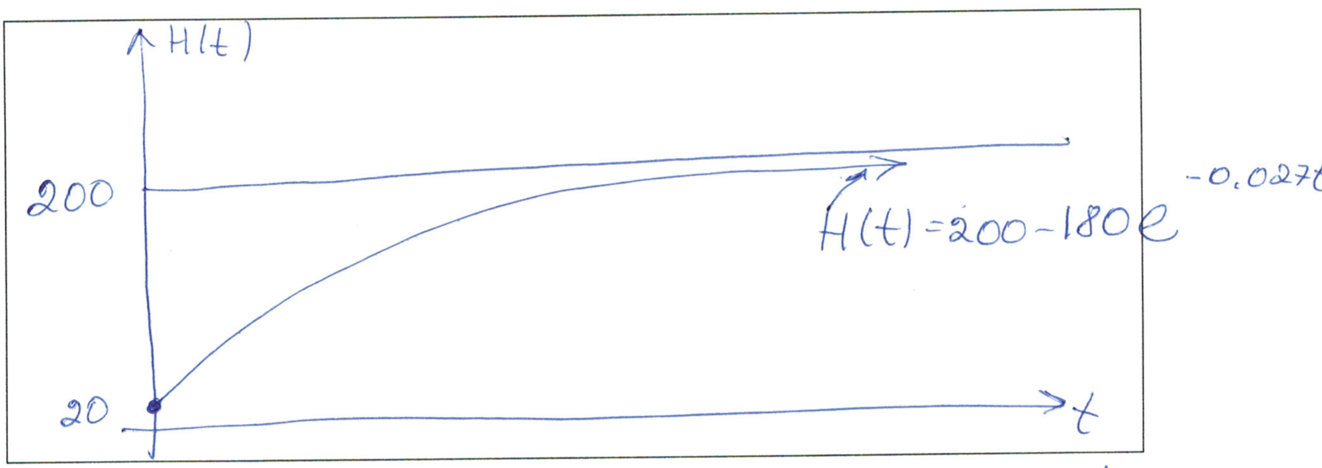
$$-30\alpha = \ln\left(\frac{4}{9}\right)$$

$$\alpha = -\frac{1}{30}\ln\left(\frac{4}{9}\right) \simeq 0.027 > 0$$

(c) Using the values from part (a) and (b), sketch the graph of the solution $H(t)$.

A =

1



the (t, H) plane

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