MAT 1332 S12 hw2

Total : 15 points

MAT 1332, Spring/Summer 2012 Assignment 2

Due May 29, 2012 at the beginning of class.

Late assignments will **not** be accepted; **nor** will unstapled assignments.

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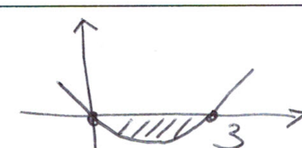
Student Name ______________________________ Student Number ______________________________

QUESTION 1. Find the area of the region bounded by the curve $y = x^2 - 3x$ and the x-axis, and determine the average value of the function $f(x) = x^2 - 3x$ on the corresponding interval. Include the sketch of the region. Leave your answers in exact form (not a decimal approximation).

$\begin{cases} y = 0 \\ y = x^2 - 3x \end{cases}$ $\quad x^2 - 3x = 0$

$x_1 = 0, \quad x_2 = 3$



1 point

$$\text{area} = \int_0^3 |x^2 - 3x|\,dx = \int_0^3 -(x^2-3x)\,dx =$$

$$= \int_0^3 (3x - x^2)\,dx = \frac{3x^2}{2}\Big|_0^3 - \frac{x^3}{3}\Big|_0^3 =$$

$$= \frac{3}{2}(9-0) - \frac{1}{3}(27-0) = \frac{27}{2} - \frac{18}{2} = \frac{9}{2}\ [\text{unit}]^2$$

1 point

$$\bar{y} = \frac{\int_0^3 (x^2-3x)\,dx}{3} = \frac{\frac{x^3}{3}\Big|_0^3 - \frac{3x^2}{2}\Big|_0^3}{3} = -\frac{3}{2}$$

$\boxed{\text{2 points}}$

QUESTION 2. Evaluate the integral $\displaystyle\int \frac{7x+5}{x^2-6x+9}dx$.

$$\int f(x)\,dx = \int \frac{P(x)}{Q(x)}\,dx, \text{ where } Q(x) = x^2 - 6x + 9 = (x-3)^2$$

$\deg(P) < \deg(Q)$

setting I

$x_1 = x_2 = 3$ (case 2).

1 point

$$f(x) = \frac{7x+5}{x^2-6x+9} = \frac{7x+5}{(x-3)^2} = \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2}$$

multipeying both sides by $(x-3)^2$, gives us

$$7x+5 = A_1(x-3) + A_2 = A_1 \cdot x + (A_2 - 3A_1)$$

Gagthering like terms, we obtain the following system

0.5

$$\begin{cases} A_1 = 7 \\ A_2 - 3A_1 = 5 \end{cases} \Rightarrow A_2 = 5 + 3A_1 = 5 + 3 \cdot 7 = 26$$

$$f(x) = \frac{7}{x-3} + \frac{26}{(x-3)^2}$$

0.5

$$\int \frac{(7x+5)dx}{x^2-6x+9} = \int \frac{7dx}{x-3} + \int \frac{26\,dx}{(x-3)^2} =$$

$$= 7\int \frac{dx}{x-3} + 26\int \frac{dx}{(x-3)^2} = 7\ln|x-3| - \frac{26}{x-3} + C.$$

$\boxed{\text{2 points}}$

② Evaluate $I = \int \frac{x^3+2x^2-18x+2}{x^2+x-12}\,dx = \int \frac{P(x)}{Q(x)}\,dx$

$\deg(P) > \deg(Q)$, then we do long division first:

$$\begin{array}{r l}
 & x+1 \\
x^2+x-12 \,\big) & x^3+2x^2-18x+2 \\
 & -\;x^3+x^2-12x \\ \hline
 & 0+x^2-6x+2 \\
 & -\;x^2+x-12 \\ \hline
 & -7x+14
\end{array}$$

Thus, $\frac{x^3+2x^2-18x+2}{x^2+x-12} = \frac{(x^2+x-12)(x+1)+(14-7x)}{x^2+x-12} =$

$= (x+1) + \frac{14-7x}{x^2+x-12}$;

1 point

$I = \int \frac{P(x)}{Q(x)}\,dx = \int \left[(x+1) + \frac{14-7x}{x^2+x-12}\right] dx =$

$= \int x\,dx + \int dx + \underbrace{\int \frac{14-7x}{x^2+x-12}\,dx}_{I_1} = \frac{x^2}{2} + x + I_1$

To evaluate I_1, use partial fraction decomposition

$\left.\begin{array}{l} x^2+x-12=0 \\ D = 1-4(-12) = 49 > 0 \\ x_1 = 3, \quad x_2 = -4. \end{array}\right\}$ Case 1.

$x^2+x-12 = (x-3)(x+4)$

Thus, $\frac{14-7x}{x^2+x-12} = \frac{14-7x}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$

$$14-7x = A(x+4)+B(x-3) = (A+B)x + (4A-3B)$$

$$\begin{cases} A+B=-7 \\ 4A-3B=14 \end{cases} \Rightarrow \boxed{A=-1}, \boxed{B=-6}$$

1 point

Therefore, $\displaystyle\int \frac{14-7x}{(x-3)(x+4)}\,dx = \int -\frac{dx}{x-3} - \int \frac{6\,dx}{x+4} =$

$= -\ln|x-3| - 6\ln|x+4| + C.$

1 point

Going back to the original integral:

$$\boxed{I = \frac{x^2}{x} + x - \ln|x-3| - 6\ln|x+4| + C}$$

← the final answer.

3 points

$\boxed{Q4}$ (a) $\int \frac{dx}{x^2+14x+55}$

$Q(x) = x^2+14x+55$

$\mathcal{D} = 196 - 4 \cdot 55 < 0 \Rightarrow$ Case 3.

0.5 point.

$$\int \frac{dx}{x^2+14x+55} = \int \frac{dx}{(x+7)^2+6} = \boxed{\begin{array}{l} x+7 = u(x) \\ dx = du \end{array}}$$

1.5 points

$$= \int \frac{du}{u^2+6} = \int \frac{du}{6\left(\frac{u^2}{6}+1\right)} =$$

$$= \frac{1}{6}\int \frac{du}{\left(\frac{u}{\sqrt{6}}\right)^2+1} = \boxed{\begin{array}{l} \frac{u}{\sqrt{6}} = z(u) \\ du = \sqrt{6}\, dz \end{array}} =$$

$$= \frac{1}{6}\int \frac{\sqrt{6}\, dz}{z^2+1} = \frac{\sqrt{6}}{6}\int \frac{dz}{z^2+1} =$$

$$= \frac{1}{\sqrt{6}} \arctan(z) + C = \frac{1}{\sqrt{6}} \arctan\left(\frac{u}{\sqrt{6}}\right) + C =$$

$$= \frac{1}{\sqrt{6}} \arctan\left(\frac{x+7}{\sqrt{6}}\right) + C.$$

$\boxed{\text{2 points}}$

(b) $$\int \frac{(x+7)\,dx}{x+14x+55} = \int \frac{(x+7)\,dx}{(x+7)^2+6}$$

$$\boxed{\begin{array}{l} x+7=u(x) \\ dx=du \end{array}} \quad = \int \frac{u\,du}{u^2+6} = \quad \boxed{\begin{array}{l} u^2+6=z(u) \\ 2u\,du=dz \\ u\,du=\frac{1}{2}dz \end{array}}$$

$$= \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln|z|+C =$$

$$= \frac{1}{2}\ln|u^2+6| + C = \frac{1}{2}\ln|x^2+14x+55|+C$$

$\boxed{\text{1 point}}$

$\boxed{\text{Example}}$ Calculate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{\sqrt{\sin x}}$ type 2.

$=$

$$\boxed{\begin{array}{l} \sin x = u(x) \\ \cos x = \dfrac{du}{dx} \Rightarrow \cos x\,dx = du \\ \begin{array}{c|c|c} x & 0 & \frac{\pi}{2} \\ \hline u & 0 & 1 \end{array} \end{array}}$$

$$= \int_0^1 \frac{du}{\sqrt{u}} \overset{\boxed{p=\frac{1}{2}}}{=} \int_0^1 u^{-\frac{1}{2}}\,du = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 u^{-\frac{1}{2}}\,du =$$ 0.5

$$= \lim_{\varepsilon \to 0^+} \left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_\varepsilon^1 \right] = \lim_{\varepsilon \to 0^+} \left[2\sqrt{u} \Big|_{u=\varepsilon}^{u=1} \right] =$$ 0.5

$$= \lim_{\varepsilon \to 0^+} \left[2\sqrt{1} - 2\sqrt{\varepsilon} \right] = 2.$$

0.5 for the definition
0.5 for the final answer $\boxed{\text{1 point}}$

$\boxed{\text{Example}}$ $\boxed{\text{1 point}}$

$$\int_{\underline{\underline{0}}}^{\infty} \frac{dx}{\sqrt[3]{x+2}} \overset{\text{type1}}{=} \lim_{T\to\infty} \int_0^T \frac{dx}{\sqrt[3]{x+2}} =$$

(shift)

$$\boxed{\begin{array}{ll} x+2 = u(x) & \begin{array}{c|c|c} x & 0 & T \\ \hline u & 2 & T+2 \end{array} \\ dx = du & \end{array}}$$

$$= \lim_{T\to\infty} \int_2^{T+2} \frac{du}{\sqrt[3]{u}} \overset{p=\frac{1}{3}}{=} \lim_{T\to\infty} \int_2^{T+2} u^{-\frac{1}{3}}\,du =$$ 0.5

$$= \lim_{T\to\infty} \left[\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \Bigg|_{u=2}^{u=T+2} \right] = \lim_{T\to\infty} \left[\frac{3u^{\frac{2}{3}}}{2} \Bigg|_{u=2}^{u=T+2} \right] =$$ 0.5

$$= \lim_{T\to\infty} \left[\frac{3}{2} \cancel{4} (T+2)^{\frac{2}{3}} - \frac{3}{2} \cdot (2)^{\frac{2}{3}} \right] \to \infty$$

the integral is divergent, $p = \frac{1}{3} < 1$ type 1.

Example

$\boxed{\text{1 point}}$

$$\int_{-2}^{2} \frac{dx}{\sqrt{2-x}} \overset{\text{type 2}}{=}$$

Substitution
$2-x=u(x)$
$-dx=dy$

x	-2	2
u	4	$0.$

0,5

$$= \int_{4}^{0} -\frac{du}{\sqrt{u}} = \int_{0}^{4} \frac{du}{\sqrt{u}} \overset{\boxed{p<1}\Rightarrow \text{convergence.}}{=}$$

$$= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{4} u^{-\frac{1}{2}}\,du = \lim_{\varepsilon \to 0^+} \left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_{\varepsilon}^{4} \right] =$$

0,5

$$= \lim_{\varepsilon \to 0^+} \left[2u^{\frac{1}{2}} \Bigg|_{u=\varepsilon}^{u=4} \right] = \lim_{\varepsilon \to 0^+} 2\left(\sqrt{4} - \sqrt{\varepsilon}\right) = 4.$$

or

$$\int_{-2}^{2} \frac{dx}{\sqrt{2-x}} = \lim_{E \to 2^-} \int_{-2}^{E} \frac{dx}{\sqrt{2-x}} =$$

$2-x=u(x)$
$-dx=du$

x	-2	E
u	4	$2-E$

$$= \lim_{E \to 2^-} \int_{4}^{2-E} -\frac{du}{\sqrt{u}} = \lim_{E \to 2^-} \int_{2-E}^{4} \frac{du}{u^{\frac{1}{2}}} =$$

$$= \lim_{E \to 2^-} \left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) \Bigg|_{u=2-E}^{u=4} = \lim_{E \to 2^-} \left(2u^{\frac{1}{2}} \right) \Bigg|_{u=2-E}^{u=4}$$

$$= \lim_{E \to 2^-} \left(2 \cdot 2 - 2 \cdot \sqrt{2-E} \right) = 4.$$

Example Evaluate $\int_a^{\infty} e^{-t}\cos t\,dt =$ type 1

$$= \lim_{T\to\infty} \int_a^{T} e^{-t}\cos t\,dt$$

First, let's evaluate the indefinite integral

$$\int e^{-t}\cos t\,dt =$$

$\cos t\,dt = dv = v'dt$; $e^{-t} = u$

$v' = \cos t$; $-e^{-t}dt = du$

$v = \sin t$

$$= e^{-t}\sin t - \int \sin t\,(-e^{-t})\,dt = e^{-t}\sin t + \int e^{-t}\sin t\,dt$$

= integration by parts one more time

$e^{-t} = u$; $-e^{-t}dt = du$

$\sin t\,dt = dv = v'dt$; $v' = \sin t$; $v = -\cos t$

$$= e^{-t}\sin t + e^{-t}\cdot(-\cos t) - \int(-\cos t)(-e^{-t}dt) =$$

$$= e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\cos t\,dt$$

Look at the LHS and at the RHS, then put $-\int e^{-t}\cos t\,dt$ to the LHS; you get

$$2\int e^{-t}\cos t\,dt = e^{-t}\sin t - e^{-t}\cos t$$

$$\int e^{-t}\cos t\,dt = \frac{e^{-t}\sin t - e^{-t}\cos t}{2}$$

Thus, $$\lim_{T\to\infty}\int_a^{T} e^{-t}\cos t\,dt = \lim_{T\to\infty}\left[\frac{e^{-t}\sin t - e^{-t}\cos t}{2}\right]\Bigg|_{t=a}^{t=T}$$

$$= \lim_{T\to\infty}\left[\frac{e^{-T}(\sin T-\cos T)}{2} - \frac{e^{-a}(\sin a-\cos a)}{2}\right] =$$

$$= \lim_{T\to\infty}\left[\frac{(\sin T-\cos T)}{2e^{T}} - \frac{(\sin a-\cos a)}{2e^{a}}\right] =$$

0 as $T\to\infty$

$$= \frac{\cos a-\sin a}{2e^{a}}.$$ ← the value of the integral; it is a finite number.

1 point for integration by parts twice

0.5 for the writing of the definition of improper integral of type 1.

0.5 for the limit and final answer.

$\boxed{2}$ points