MAT 1330, Fall 2011 Assignment 4

Due November 9 at the beginning of class.

Late assignments will **not** be accepted; **nor** will unstapled assignments.

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Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. Find the derivatives of the following functions. Do not simplify your answers.

(a) $f(x) = \cos(3x)\sin(7x^3 + 2x)$

$f'(x) =$ $-3\sin(3x)\cdot\sin(7x^3+2x) + \cos(3x)\cdot\cos(7x^3+2x)\cdot(21x^2+2)$

(b) $g(x) = \dfrac{\tan(x^2)}{e^{5x}+1}$; $g'(x) = \dfrac{(\tan(x^2))'\cdot(e^{5x}+1) - \tan(x^2)\cdot(e^{5x}+1)'}{(e^{5x}+1)^2}$

$g'(x) =$ $\dfrac{\frac{1}{\cos^2(x^2)}\cdot 2x\cdot(e^{5x}+1) - \tan(x^2)\cdot 5e^{5x}}{(e^{5x}+1)^2}$

(c) $h(x) = e^{\cos^2(x)+3\sin(x^3)}$

$h'(x) =$ $e^{\cos^2(x)+3\sin(x^3)}(2\cos(x)(-\sin x) + 3\cos(x^3)\cdot 3x^2)$

(d) $w(x) = \sqrt{x + 3\ln x + 4}$

$w'(x) =$ $\dfrac{1}{2\sqrt{x+3\ln x+4}}\cdot\left(1 + 3\cdot\frac{1}{x}\right)$

QUESTION 2. Find the global minimum and the global maximum of $f(x) = x - \ln x$ on the interval $[0.1, 2]$.

$f'(x) = 1 - \frac{1}{x}$

$f'(x) = 0 \qquad 1 - \frac{1}{x} = 0$

$x = 1$ (critical point)

$f(0.1) = 0.1 - \ln(0.1) \simeq 2.4025 \leftarrow$ value of f at the endpoint $x = 0.1$

$f(1) = 1 - \ln(1) = 1$

$f(2) = 2 - \ln(2) \approx 1.3068 \leftarrow$ value of f at the endpoint $x = 2$.

Global maximum $\boxed{2.4}$ at $x = \boxed{0.1}$.

Global minimum $\boxed{1}$ at $x = \boxed{1}$.

QUESTION 3.

Consider the following DTDS:

$$x_{t+1} = \frac{1.5x_t^2}{x_t^2 + 0.5}.$$

(a) The DTDS has three equilibrium points. Give the equilibrium points in increasing order.

$$f(X) = \frac{1.5X^2}{X^2+0.5} = X$$

$$\frac{1.5X^2}{X^2+0.5} - X = 0$$

$$\frac{1.5X^2 - X^3 - 0.5X}{X^2+0.5} = 0 \iff 1.5X^2 - X^3 - 0.5X = 0 \quad \text{or}$$

$$X(1.5X - X^2 - 0.5) = 0$$

$$X_1^* = 0$$

$$X^2 - 1.5X + 0.5 = 0$$

$$\Delta = 2.25 - 4 \cdot 0.5 = 0.25$$

$$X_2^* = \frac{1.5-0.5}{2} = 0.5 \qquad X_3^* = \frac{1.5+0.5}{2} = 1$$

$x_1^* =$ $\boxed{0}$

$x_2^* =$ $\boxed{0.5}$

$x_3^* =$ $\boxed{1}$

(b) Find the derivative of the updating function of the DTDS.

$f'(x) =$ $\boxed{\frac{1.5X}{(X^2+0.5)^2}}$

$$f'(X) = \frac{1.5 \cdot 2X(X^2+0.5) - 1.5X^2 \cdot 2X}{(X^2+0.5)^2} = \frac{3X^3 + 1.5X - 3X^3}{(X^2+0.5)^2} =$$

$$= \frac{1.5X}{(X^2+0.5)^2}$$

(c) For each of the equilibrium points in (a) find $f'(x_i^*)$ and decide whether the equilibrium point is stable (circle the correct answer).

$f'(x_1^*) = \boxed{0}$ stable unstable

$f'(x_2^*) = \boxed{\frac{1.5 \cdot 0.5}{(0.25+0.5)^2} \approx 1.33}$ stable unstable

$f'(x_3^*) = \boxed{\frac{1.5}{2.25} \approx 0.6}$ stable unstable

Question 4.

The number of individuals (in thousands) of a certain species satisfies the DTDS:

$$x_{t+1} = \frac{4x_t}{1+x_t} - hx_t, \quad t = 0, 1, 2, \ldots$$

The population is harvested with the rate $h \geq 0$. Answer the following questions:

(a) The equilibrium points of this DTDS are (Hint: one of the equilibrium points will depend on h):

$\boxed{0}$ and $\boxed{\frac{3-h}{1+h}}$

$f(x) = x$

$\frac{4x}{1+x} - hx = x$

$\frac{4x}{1+x} - hx - x = 0$

$x\left[\frac{4}{1+x} - h - 1\right] = 0$

$x_1^* = 0$

$\frac{4}{1+x} = h+1$

$(1+x)(h+1) = 4$

$1+x = \frac{4}{h+1}$

$x = \frac{4}{h+1} - 1 = \frac{4-h-1}{h+1} = \frac{3-h}{1+h}$

$x_2^* = \frac{3-h}{1+h}$

(b) Give the largest interval for h such that both equilibrium points in (a) are non-negative (i.e. biologically meaningful).

$h \in \boxed{[0, 3]}$

(c) What is the harvest at the positive equilibrium? (Hint: the answer will depend on h)

$R(h) = \boxed{h \cdot \frac{(3-h)}{1+h}}$

(d) The maximal harvest at the positive equilibrium is

$R'(h) = 0 \quad \left(\frac{3h-h^2}{1+h}\right)' = \frac{(3-2h)(1+h) - (3h-h^2)\cdot 1}{(1+h)^2} = \frac{3-2h+3h-2h^2-3h+h^2}{(1+h)^2}$

$= \frac{-h^2-2h+3}{(1+h)^2} = 0$ or $h^2+2h-3=0$

$D = 4 - 4(-3) = 16$

$h_1 = \frac{-2+4}{2} = 1$

$h_2 = \frac{-2-4}{2} = -3$ ← exclude, since $h \geq 0$.

$R(0) = R(3) = 0$

$\boxed{R(1) = \frac{1 \cdot (3-1)}{1+1} = 1}$ ← glob. max

$R_{max} =$ $\boxed{1}$ and occurs for the harvesting rate $h =$ $\boxed{1}$

(e) For the values of h as in (b) let x^* be the positive equilibrium point. Then:

x^* is stable when h is $\boxed{[0,3)}$

x^* is unstable when h is $\boxed{\text{never}}$

$$f'(x) = \left(\frac{4x}{1+x}\right)' - h = \frac{4\cdot(1+x) - 4x\cdot 1}{(1+x)^2} - h = \frac{4}{(1+x)^2} - h$$

$$\left|f'\left(\frac{3-h}{1+h}\right)\right| = \left|\frac{4}{\left(1+\frac{3-h}{1+h}\right)^2} - h\right| = \left|\frac{(1+h)^2}{4} - h\right| < 1$$

$$\Rightarrow |1+2h+h^2-4h| < 4 \quad \text{or} \quad |1-2h+h^2| < 4$$

$$|(h-1)^2| < 4$$

$$(h-1)^2 < 4$$

$$|h-1| < 2$$

$$-2 < h-1 < 2$$

$$-1 < h < 3$$

$$h \geq 0$$

Thus, if $h \in [0,3)$ then x_2^* is stable. Otherwise, it is unstable, but x_2^* only exists if $h \in [0,3)$. Therefore, x_2^* is never unstable.