H_O : $\mu_1 = \mu_2 = \mu_3 = \ldots \mu_k$ or $\mu_1 - \mu_2 = 0$
H_a : at least one μ is $\neq$ to the others or $\mu_1 - \mu_2 \neq 0$
k treatment groups

ONE WAY ANOVA

Definition of CRD: A random selection of k independent samples (treatments) from k populations

ASSUMPTIONS FOR CRD:

1) k independent random samples (random selection) of sizes n_1, n_2, …. to n_k
2) k independently NORMALLY DISTRIBUTED populations with means μ_1, μ_2, μ_3…. μ_k
3) and…. EQUAL VARIANCE, σ^2 (**MUST test Assumption of Equal Variance with HARTLEY'S TEST FIRST**)

Samples	Observations	Treatment Totals	Treatment Means
1	$y_{11}, y_{12}, y_{13}, \ldots. y_{1n}$	$T_1 = \sum_{j=1}^{n_1} y_{1j}$	$\overline{T_1} = \frac{T_1}{n_1}$
2	$y_{21}, y_{22}, y_{23}, \ldots. y_{2n}$	$T_2 = \sum_{j=1}^{n_2} y_{2j}$	$\overline{T_2} = \frac{T_2}{n_2}$
…			
k	$y_{k1}, y_{k2}, y_{k3}, \ldots. y_{kn}$	$T_2 = \sum_{j=1}^{n_k} y_{kj}$	$\overline{T_k} = \frac{T_k}{n_k}$

N = $\sum_{i=1}^{k} n_i$ $\bar{y}. = \overline{T}_i$

i =counts # of treatments $\boldsymbol{i = 1 \ldots k}$
j = counts number of observations for treatment i $\boldsymbol{j = 1 \ldots n_i}$

G.T. = GRAND TOTAL = Sum of All Treatment Totals across all treatments

G.T. = $T_1 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}$

Overall Mean, $\bar{Y}$ (or $\bar{y}_{..}$) = $\frac{\text{GT}}{\text{N}}$

TSS = SSTr + SSE

TSS = $\sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2$ - Measures TOTAL VARIATION of y's around their mean i.e. VAR(y)

SSTr = $\sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{y}_{.} - \bar{y}_{..})^2$ - Measures how treatment means ($\bar{y}_{.}$) vary around the overall mean ($\bar{y}_{..}$)

SSE= $\sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{.})^2$ - Measures the variation of y's around their sample mean ($\bar{y}_{i.}$) (i.e. measures variation WITHIN sample)

COMPUTATIONAL FORMULAS:

$$\text{TSS} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} {y_{ij}}^2 - \frac{\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}\right)^2}{N}$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} {y_{ij}}^2 - \frac{\text{G.T.}}{\text{N}}$$

$\frac{\text{G.T.}}{\text{N}}$ = Correction for the Mean (C.M.)

$$\text{SSTr} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{T_i^2}{n_i} - \text{C.M.}$$

SSE= TSS - SSTr

df_{TSS} = N-1
df_{SSTr} = k-1
df_{SSE} = (N-1) –(k-1) = N-K

MSTr = $\frac{\text{SSTr}}{\text{k}-1}$ MSE = $\frac{\text{SSE}}{\text{N}-\text{k}}$ $F_T = \frac{\text{MSTr}}{\text{MSE}}$

ANOVA

Source of Var.	df	SS	MS	F
Treatments	k-1	SSTr	MSTr	MSTr/MSE
ERROR	N-k	SSE	MSE	
Total	N-1	TSS		

Reject Ho if value of $F_T > F_{\alpha(k-1,\, n-k)}$

CRD is appropriate when:

1) The design uses appropriate homogenous experimental units
2) experimental units are randomly assigned to k different treatments
3) k treatments are INDEPENDENT

Must TEST EQUALITY OF VARIANCE before testing MODEL using HARTLEY'S TEST

HARTLEY'S TEST for equality of treatment variances

$H_O : \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \ldots \sigma_k^2$
H_a : at least one σ^2 is $\neq$ to the others

Note: Hartley's tables only have values for α = 1% or α = 5%
Calculate variance (S^2) for each treatment (SEE BELOW)

Test statistic: $F_{max} = \frac{S^2_{max}}{S^2_{min}}$

R.R. – Reject Ho if $F_{max} > F_{critical}$ from HARTLEY'S TABLES: $F_{\alpha;(k,[\bar{n}]-1)}$
where $[\bar{n}]$ = integer part of average sample size (i.e. if $\bar{n}$ = 5.64 then $[\bar{n}]$ = 5 – DON'T ROUND)

IF H_o is REJECTECT (i.e. NON-EQUAL VARIANCE) then STOP
IF H_o is NOT REJECTED (I.E. EQUAL VARIANCE) **proceed with main test**.

CONCLUSION to proceed: Given that $F_{max} \ngtr F_{critical}$ we can conclude, at **α** level of significance that there is **insufficient evidence** to show the variances are not equal, therefore we can ***proceed with the assumption that variances are equal***….

EX. 4 Groups of unequal sizes with different teaching techniques, where students are RANDOMLY ASSIGNED to one of the 4 groups. Do the data present sufficient evidence to indicate a difference in the mean achievement for students taught using the 4 techniques?

Teaching Technique	**Observations**	n_i	T_i	$\overline{T_i}$
1	65,87,73,79,81,69	6	454	75.67
2	75,69,83,81,72,79,90	7	549	78.43
3	59,78,67,62,83,76	6	425	70.83
4	94,89,80,88	4	351	87.75
		N=23 $[\bar{n}] = 5$	1779	

where $\overline{T_i} = \overline{y_i}$

$$S^2_{Ti} = \frac{\sum_{j=1}^{n_i} y^2_{1j} - \frac{\left(\sum_{j=1}^{n_i} y_{ij}\right)^2}{n_i}}{n_i - 1}$$ (Calculate for each treatment option)

FOLLOW-UP ANALYSIS – Multiple Comparisons

TUKEY'S HSD — to be completed after processing to determine WHERE a difference exists – USE THIS AS DEFAULT unless otherwise told to use something else.

H_O : $\mu_i = \mu_j$
H_a : $\mu_i \neq \mu_j$

IMPORTANT : EQ SHEET does not show N-k! Must remember this for df.

1) **calculate N = $n_1 + n_2 + n_3 + n_4 = 23$; k = 4**

2) **calculate $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs of $|\bar{y}_i - \bar{y}_j|$**

3) **calculate CRITICAL REGION: $hsd = q_{\alpha(k,N-k)}\sqrt{\frac{MSE}{2}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$ for $i, j = 1 \ldots k; i \neq j$**

   Where $q_{\alpha(k,N-k)}$ IS A CRITICAL VALUE FROM THE Student Range Distribution Tables, and α can only be 1% or 5%

   eg from previous example: $q_{\alpha(k,N-k)}$, where α=0.10, (use SAS output because tables only show .01 and .05) $q_{0.10(4,19)}$ = 3.475

   $|\bar{y}_1 - \bar{y}_2| = |75.67 - 78.43| = 2.76 \ngtr 3.475\sqrt{\frac{62.98058}{2}\left(\frac{1}{6}+\frac{1}{7}\right)} = 10.84889$

   ...

   $|\bar{y}_3 - \bar{y}_4| = |70.83 - 87.85| = 16.92 > 3.475\sqrt{\frac{62.98058}{2}\left(\frac{1}{6}+\frac{1}{4}\right)} = 12.58382$

4) **REJECT H_O for every instance where $|\bar{y}_i - \bar{y}_j| > hsd$:**

   Conclusion: At a 10% level of significance we can reject H_o for the teaching techniques 3 & 4 and conclude there is evidence to suggest a there is a difference between these two techniques.

FISHER'S LSD – DON'T QUOTE SIGNIFICANCE LEVEL?

H_O : $\mu_i = \mu_j$
H_a : $\mu_i \neq \mu_j$

T-TEST – must divide alpha level by 2!

1) **calculate $N = n_1 + n_2 + n_3 + n_4 = 23$; $k = 4$**

2) **calculate $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs of $\left|\bar{y}_i - \bar{y}_j\right|$**
3) **calculate CRITICAL REGION:**

$$l.s.d = t_{(N-k);\alpha/2}\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} \text{ for } i, j = 1 \dots k; i \neq j$$

4) **REJECT H_O for every instance where $\left|\bar{y}_i - \bar{y}_j\right| > l.s.d$:**

NOTE: The overall significance level of simultaneous tests is less than the original alpha level. I.e. the alpha level of each individual test ≤ $\frac{\alpha}{\binom{k}{2}} = \alpha^*$ because it is a case of BOOL'S INEQUALITY - in other words, don't QUOTE significance level in conclusion.

KRUSKAL-WALLIS TEST

BASED on RANKING from smallest to largest and taking MEDIANS.

H_O : md_1 = md_2 = md_3 = … md_k
H_a : at least one md is ≠ to the others @ α level of significance

1) STATE ASSUMPTIONS:
   a. **CRD**
   b. **Treatment populations have approx.. same shame and spread**

2) calculate N = $n_1 + n_2 + n_3 + n_4$ = 6+7+6+4 = 23; k = 4
3) RANK ALL observations from lowest to highest. Assign an AVERAGE rank to any tied scores, eg SCORES: 2 5 6 6 6 10 31
RANKS: (1) (2) (4) (4) (4) (6) (7)
(3 + 4 + 5)/3 =4

	I	ranks	II	ranks	III	ranks	IV	ranks
	65	(3)	75	(9)	59	(1)	94	(23)
	87	(19)	69	(5.5)	78	(11)	89	(21)
	73	(8)	83	(17.5)	67	(4)	80	(14)
	79	(12.5)	81	(15.5)	62	(2)	88	(20)
	81	(15.5)	72	(7)	83	(17.5)		
	69	(5.5)	79	(12.5)	76	(10)		
			90	(22)				
Medians:	**Tr_1=**	**63.5**	**Tr_2=**	**89**	**Tr_3=**	**45.5**	**Tr_4=**	**78**

4) Total up the RANKS for each TREATMENT GROUP (Tr_i)
5) CHECK that SUM of all $Tr_i = \frac{N(N+1)}{2}$ eg. 63.5 + 89 + 45.5 + 78 = 276; 23(24)/2 = 276 ✓
6) calculate the test statistic:

$$H = \frac{12}{N(N+1)}\left[\sum_{i=1}^{k}\frac{Tr_i}{n_i}\right] - 3(N+1)$$

Where Tr_i = total of ranks on treatment i
n_i = sample size of treatment i
N = grand total of all ranks

7) R.R. – we reject H_O if H > $\chi^2_{(k-1);\alpha}$
8) STATE CONCLUSION: I.E. Since H = (value) > $\chi^2_{(k-1);\alpha}$ = value, we can reject Ho AND conclude at a 10% level of significance that there is evidence of significant differences among teaching techniques.

DUNN'S PROCEDURE – POST HOC FOLLOWUP TEST FOR NON-PARAMETRIC DISTRIBUTION

1) **Determine which medians are different: Use Tr_i taken from previous ranking exercise for Kruskal-Wallis test**

2) **calculate $\binom{k}{2}$ pairs of $|\overline{R_i} - \overline{R_j}|$ where $\overline{R_i} = \frac{Tr_i}{n_i}$**

3) **Obtain critical Z from UPPER TAIL Z tables.**
   i.e. if α = 0.10 and k=4 then:

   $\frac{\alpha}{k(k-1)} = \frac{0.1}{4(4-1)} = 0.008333$ **FROM THE TABLES: Z (1-0.00833) = 2.395**

4) **For each comparison: H_o: $Md_i = Md_j$; H_a: $Md_i \neq Md_j$; where i,j = 1…k & i ≠ j**

5) **Calc Critical region. for each comparison: C.R. = $Z_{\frac{\alpha}{k(k-1)}}\left(\sqrt{\frac{N(N+1)}{12}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}\right)$**

6) Reject H_o for any given pair tested if $|\overline{R_i} - \overline{R_j}|$ **> C.R.**

7) **State CONCLUSION:** i.e.
   There is evidence to suggest that there are differences between training programs 3 and 4

2-WAY ANOVA TESTING

If experiments are NOT HOMOGENOUS then we can divide it into homogenous groups called BLOCKS
WITHIN BLOCK is HOMOGENOUS but each block may differ from the next
ALL treatments are RANDOMLY ASSIGNE within each block, each block must have ALL treatments and treatments must appear the same # of times in each block

Definition (for this class)
A RANDOMIZED BLOCK DESIGN is an EXPERIMENTAL DESIGN for comparing "k" treatments in "b" blocks. Treatments are randomly assigned to experimental units with each block, with each treatment appearing exactly ONCE within every block.

Model: $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$; where $i = 1 \ldots k$ & $j = 1 \ldots b$
α_i = treatment effect
β_j = block effect
y_{ij} = observation in the i^{th} treatment and the j^{th} block

	Treatments					
Blocks	**1**	**2**	**j**	**b**	Block totals	Block Means
1	y_{11}	y_{12}	y_{1j}	y_{1b}	B_1	B_1/b
2	y_{21}	y_{22}	y_{2j}	y_{2b}	B_2	B_2/b
….						
k	y_{k1}	y_{k2}	y_{kj}	y_{kb}	B_b	B_b/b
Treatment Totals	Tr_1	Tr_2	Tr_j	Tr_b	**GT**	
Treatment Means	Tr_1/b	Tr_2/b	Tr_3/b	Tr_b/b		

Where **GT** = sum of all Treatment Totals = Sum of all Block totals
$\bar{y}_{..} = \frac{GT}{bk}$
sample size for treatments = **b** (across ALL treatments)
sample size for blocks = **k** (across all blocks)

TSS = Variance of y values $= \sum_{i=1}^{k} \sum_{j=1}^{b} y_{ij}^2 - \frac{GT^2}{bk}$

SSTr = Variation BETWEEN treatment means ($\overline{y_{i.}}$) $= \sum_{i=1}^{k} \frac{T_i^2}{b} - \frac{GT^2}{bk}$

SSB = Variation of BLOCK means ($\overline{y_{.j}}$) around the OVERALL mean = $\sum_{i=1}^{k} \frac{B_j^2}{k} - \frac{GT^2}{bk}$

SSE = Variation due to error = **TSS – SSTr – SSB**

df_{TSS} = N-1 = bk-1
df_{SSTr} = k-1
df_{SSB} = b-1
df_{SSE} = (b-1)(k-1) → only if treatment appears ONCE in each block

ANOVA

Source of Var.	df	SS	MS	F
Treatments	k-1	SSTr	MSTr	F_T = MSTr/MSE
Blocks	b-1	SSB	MSB	F_B = MSB/MSE
ERROR	N-k	SSE	MSE	
Total	N-1	TSS		

STEPS in RBD analysis:

1) STATE ASSUMPTIONS:

1. k treatments are RANDOMLY ASSIGNED within each block so that each treatment occurs EXACTLY ONCE in every block and every block contains ALL TREATMENTS
2. Normality: Populations corresponding to the treatment block combinations have NORMAL DISTRIBUTION with…
3. EQUAL VARIANCE of σ^2 (MUST be CONFIRMED using HARTLEY'S TEST*)
4. BLOCKS and TREATMENTS do NOT interact.

* Can only proceed with main test if variances are shown to be equal via Hartley's Test

2) Run HARTLEY'S test for equal variance:

$H_O : \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \ldots \sigma_k^2$; α= 0.01 or 0.05
H_a : at least one σ^2 is ≠ to the others

Calculate S^2 for all treatments:

$$S_A^2 = \frac{\sum_{j=1}^{b} y_{Aj}^2 - \frac{\left(\sum_{j=1}^{b} y_{Aj}\right)^2}{b}}{b-1}$$ (for each treatment) **Test statistic:** $\boldsymbol{F_{max} = \frac{S_{max}^2}{S_{min}^2}}$

R.R. – Reject Ho if $F_{max} > F_{critical}$ from HARTLEY'S TABLES: $F_{\alpha;(k,[\bar{n}]-1)}$
where $[\bar{n}]$ = integer part of average sample size – in the case of RBD , since sample size is always "b" for all treatments, $[\bar{n}]$ = b

IF H_o is REJECTECT (i.e. NON-EQUAL VARIANCE) then STOP
IF H_o is NOT REJECTED (I.E. EQUAL VARIANCE) **proceed with main test**.

CONCLUSION to proceed: Given that $F_{max} \not> F_{critical}$ we can conclude, at **α** level of significance that there is **insufficient evidence** to show the variances are not equal, therefore we can ***proceed with the assumption that variances are equal***….

3) **STATE the Null and Alternative Hypotheses for the study:**
   $H_O : \mu_1 = \mu_2 = \mu_3 = \ldots \mu_k$; α = alpha level
   H_a : at least one μ is $\neq$ to the others

4) **Test for differences in TREATMENT MEANS**
   (Calculate TSS, SSTr, SSB, SSE, F_T and F_B and put in ANOVA table.)

5) **Write conclusion with respect to TREATMENT MEANS:**
   i.e.
   Since $F_T > F_{(k-1,(b-1)(k-1));\alpha}$ we can conclude at a (α*100)% level of significance that there is evidence that the (block type) between the (Treatments) differs

6) if we REJECT H_o → Must **follow up with PARAMETRIC TEST: Tukey's HSD,** and **NON-PARAMETRIC TEST (Friedman RANK TEST)** to determine where the differences are.

E.G. Want to test which is the best of 3 courier companies. Will measure delivery time at 4 different times during the day.
COURIERS are TREATMENTS → TREATMENTS are always what we are ultimately trying to measure the difference between
TIMES OF DAY are BLOCKS → blocks are different paradigms in which to measure the treatment effects

	Couriers (Treatments)				
Times of day (Blocks)	**A**	**B**	**C**	Block totals	Block Means
9:30am	3.6	4.2	5.0	**12.8**	4.26667
11:30am	5.4	5.8	7.0	**18.2**	6.06667
12:30pm	6.1	7.0	9.1	**22.2**	7.4
2:00pm	3.5	4.0	4.9	**12.4**	4.13333
Treatment Totals	**18.6**	**12.0**	**26.0**	**GT = 65.6**	
Treatment Means	**4.65**	**5.25**	**6.5**		

N = b*k = 4*3 = 12
k = 3
b = 4

TUKEY'S HSD for RBD

H$_O$: $\mu_i = \mu_j$
H$_a$: $\mu_i \neq \mu_j$

IMPORTANT : EQ SHEET does not show N-k! Must remember this for df.

1) **calculate N = $n_1 + n_2 + n_3 + n_4 = 23$; k = 4**
2) **calculate** $\binom{k}{2} = \frac{k(k-1)}{2}$ **pairs of** $\left|\bar{y}_i - \bar{y}_j\right|$
3) **calculate CRITICAL REGION:**
4) $hsd = q_{(k,(b-1)(k-1));\,\alpha}\sqrt{\frac{MSE}{2}\left(\frac{1}{b}+\frac{1}{b}\right)}$ **=** $q_{(k,(b-1)(k-1));\,\alpha}\sqrt{\frac{MSE}{b}}$

   Where $q_{(k,(b-1)(k-1));\,\alpha}$ **IS A CRITICAL VALUE FROM THE Student Range Distribution Tables, and α can only be 1% or 5%**

5) **RE**JECT H$_O$ for every instance where $\left|\bar{y}_i - \bar{y}_j\right| > hsd$:

   Conclusion: **At a (α*100)% level of significance we can reject H$_o$ for the (treatments) (*treatment pairs*) and conclude there is evidence to suggest a there is a difference between these (treatments).**

FRIEDMAN RANK TEST – like KW – uses ranking – tests TREATMENT MEDIANS

H_O : $md_1 = md_2 = md_3 = \ldots md_k$
H_a : at least one md is ≠ to the others @ α level of significance

1) **STATE ASSUMPTIONS:**
   1. RBD
   2. Each group/block (name them) is distributed with approximately the same shape and spread
   3. (Blocks) and (treatments) do NOT interact

2) **Rank observations by BLOCK – (each block is independently ranked)**
3) **Sum ranks for each treatment:**

	Couriers (Treatments)					
Times of day (Blocks)	A	rank	B	rank	C	rank
9:30am	3.6	(1)	4.2	(2)	5.0	(3)
11:30am	5.4	(1)	5.8	(2)	7.0	(3)
12:30pm	6.1	(1)	7.0	(2)	9.1	(3)
2:00pm	3.5	(1)	4.0	(2)	4.9	(3)
Treatment Totals (rank)		4		8		12
Treatment Means						

4) **CHECK: Sum of Treatment ranks** = $\frac{bk(k+1)}{2}$

   4 + 8 + 12 = 24

   $\frac{4*3(3+1)}{2}$ = 24 ✓

5) Calculate TEST STATISTIC $F_r = \left(\frac{12}{bk(k+1)}\left[\sum_{i=1}^{k} Tr_i^2\right]\right) - 3b(k+1)$

6) **ie.** $F_r = \left(\frac{12}{12(4)}\left[4^2 + 8^2 + 12^2\right]\right) - 3(4)(4)$ **= 8**

7) **Compare to Chi-squared value:** $X^2{}_{\alpha;(k-1)} = X^2{}_{0.01;(2)} = 9.210$

8) Write conclusion:
9) Since F_r **= 8** $\ngtr$ $X^2{}_{0.01;(2)} = 9.210$ **we DO NOT reject H_O and conclude that at a 1% level of significance, we do not have enough evidence to say that there is a difference in delivery times between couriers.**

NEMENYI'S PROCEDURE

Also tests Treatment MEDIANS

1) **Calculate** $\binom{\boldsymbol{k}}{\boldsymbol{2}} = \frac{\boldsymbol{k(k-1)}}{\boldsymbol{2}}$ **pairs of** $|\overline{\boldsymbol{R_i}} - \overline{\boldsymbol{R_j}}|$
   where H_O **:** $md_i = md_j$ **;**where i, j = 1…k & i ≠ j
   H_a **:** $md_i \neq md_j$ **@ α level of significance**

   $\boldsymbol{k} = \#\ of\ treatment\ groups;\quad \boldsymbol{R}_i = \frac{\boldsymbol{Tr_i}}{\boldsymbol{b}}$;

2) **Calculate CRITICAL REGION:** $CR = Z_{\alpha(k,\infty)}\left(\sqrt{\frac{k(k+1)}{12b}}\right)$ → only need to calc once!

   Critical Value from Studentized-Range distribution tables

3) REJECT H_O **for any where that** $|\overline{\boldsymbol{R_i}} - \overline{\boldsymbol{R_j}}|$ **>** CR
4) State CONCLUSION: Based on Nemenyi's procedure, at a significance level of (α*100)%, there are differences between (Tr_x) and (Tr_y)