1. Let S be a subset of R which is bounded above and denote $\alpha = \sup S$.
   (a) Prove that for any $\varepsilon > 0$, there exists $s \in S$ such that $\alpha - \varepsilon < s \leq \alpha$.
   (b) Prove that there exists a sequence $\{s_n\}_{n=1}^{\infty} \subset S$ such that $s_n \to \alpha$.

2. (a) Prove that $f(x) = 2x + 5$ is uniformly continuous on R.
   (b) Suppose that a function f is uniformly continuous on an interval D. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in D. Prove that the sequence $\{f(x_n\}_{n=1}^{\infty}$ is Cauchy.
   (c) Prove that $f(x) = \frac{1}{x^4}$ is not uniformly continuous on $(0, 1]$.
   (d) Prove that $f(x) = \frac{1}{x^4}$ is uniformly continuous on $[1.\infty)$.
   (e) TRUE OR FALSE: If a function is continuous on $(0, 3)$, then it is uniformly continuous on $[1, 2]$. Explain briefly or provide a counterexample.
   (f) TRUE OR FALSE: If a function is uniformly continuous on $[0, 3]$, then it is uniformly continuous on $(1, 2)$. Explain briefly or provide a counterexample.

3. (a) Suppose that $f : R \to R$ is continuous, and that $\{x_n\}_{n=1}^{\infty}$ is a sequence in the interval $[a, b]$. Prove that if x_n converges to x, then $x \in [a, b]$.
   (b) Suppose that $f : R \to R$ is continuous at c, and that $f(c) > 0$. Prove that there is a neighborhood U of c such that $f(x) > 0$ for all $x \in U$.
   (c) Use the Intermediate Value Theorem to show that the function $f(x) = 3x^4 + x - 7$ has a root in the interval $[1, 2]$.

4. (a) Suppose that S is an infinite subset of R. Prove that S contains a sequence of distinct points.
   (b) Suppose that S is a subset of R which is unbounded above. Prove that S contains a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to \infty$.
   (c) Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded above. Prove that it has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \infty$.

5. (a) Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence which does not converge to x. Prove that there exists $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\|x_{n_k} - x\| \geq \varepsilon$ for all k in N.
   (b) Also show that there are at least two subsequences of $\{x_n\}_{n=1}^{\infty}$ with distinct limits.

6. Use induction to prove that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

7. Let $f : \mathcal{R} \to \mathcal{R}$ be given by

$$f(x) = \begin{cases} x^4 \sin(\frac{5}{x^2+x^3}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Prove that f is differentiable at $x = 0$ and find $f'(0)$.

8. Find the Taylor polynomial $P_3(x)$ for the function $f(x) = x^{1/2}$ about the point $x = 1$ and state the formula for the the remainder, $R_3(x)$.