MATH 364
Sample Examination
December 2006

1 a) *(7 marks)* Let $\phi \neq A \subset B \subset \mathbb{R}$. Prove that

$$\inf B \le \inf A \le \sup A \le \sup B$$

b) *(3 marks)* Give an example of two sets $A, B \subset \mathbb{R}$ such that $\inf B < \inf A < \sup A < \sup B$ but $A \cap B = \phi$.

2 a) *(7 marks)* Let S be a set of all open intervals (m, n) where $m, n \in \mathbb{Z}$ $(m < n)$. Prove that the set is countable.

b) *(6 marks)* Prove that the set $T = [0,1] \setminus \{\frac{1}{n} : n \in n\}$ is uncountable.

3 a) *(5 marks)* Let $a_n = \dfrac{1}{n}\cos n$ and let $(b_n)_{n\in\mathbb{N}}$ be bounded. Prove that the sequence $(c_n)_{n\in\mathbb{N}}$, where $c_n = a_n b_n$, converges to 0.

b) *(3 marks)* Give an example of two divergent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}(a_n + b_n) = 1$.

c) *(5 marks)* Use the Squeeze Theorem to find the limit of the sequence

$$b_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$$

[Hint: consider the sequences obtained by repeatedly adding the smallest term of b_n, and then the largest term of b_n].

4 a) *(3 marks)* State the definitions of: a sequence that is bounded above, and a sequence that is bounded below.

b) *(3 marks)* What can you say about the set E of all subsequential limits of a sequence $(a_n)_{n\in\mathbb{N}}$ if $m \le a_n \le M$ $\forall n \in \mathbb{N}$? Explain.

c) *(6 marks)* Prove that if a sequence $(a_n)_{n\in\mathbb{N}}$ is not bounded below, it has a decreasing subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} a_{n_k} = -\infty$$

d) *(4 marks)* Find the limit superior and limit inferior of the sequence

$$a_n = (-1)^n + \frac{n^2-1}{n^2+1}$$

5 a) *(5 marks)* Prove directly from the definition that if $\lim_{x \to a} f(x) = L$ then $\lim_{x \to a} 4f(x) = 4L$

b) *(6 marks)* Let $\lim_{x \to c} f(x) = 2$. Prove that there is a neighbourhood U of c such that $f(x) > 1 \; \forall x \in U \setminus \{c\}$

6 a) *(5 marks)* State the definition of continuity of a real-valued function f at a point a

(i) using the $\epsilon - \delta$ formulation

(ii) using sequences

b) *(5 marks)* Consider the function

$$f(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Can you find real number(s) where f is continuous? Explain.

c) *(7 marks)* Let $f : [0,1] \to [0,1]$ be continuous. Prove that the equation

$$f(x) = x^2$$

has a solution in $[0,1]$ (you may use the Intermediate Value Theorem).

7 Use the definition of uniform continuity to prove that for any $a > 0$

a) *(4 marks)* $f(x) = x^2$ is uniformly continuouis on $(0, a]$.

b) *(4 marks)* $f(x) = x^{-2}$ is not uniformly continuouis on $(0, a]$.

8 a) *(7 marks)* Prove directly from the definition of the derivative that the function

$$f(x) = |x^3|$$

is differentiable on R. Find $f'(x)$. What can you say about $f''(x)$?

b) *(5 marks)* Use the Mean Value Theorem to prove the inequality

$$\ln(1 + x) < x; \forall x > 0$$

Bonus Question *(5 marks)*

Prove that the two definitions of continuity of a function f at a point a (i) and (ii) in question 6 a) are equivalent.