Sample 364

1. (a) $A \neq \emptyset \Rightarrow \inf A \leq \sup A$ (*)

important since $\sup \emptyset = -\infty$ & $\inf \emptyset = +\infty$

$\sup B$ is an upper bound of B but since $A \subset B$ it is also an upper bound of A.

$\sup A$ is the least upper bound of A i.e. $\forall \gamma$ an upper bound of A $\sup A \leq \gamma$

Taking $\gamma = \sup B$ we have: $\sup A \leq \sup B$ (**)

Similarily, $\inf B$ is a lower bound of B but since $A \subset B$ it is also a lower — " — of A.

$\inf A$ is the greatest lower bound of A i.e. $\forall \gamma$ a lower bound of A $\gamma \leq \inf A$

Taking $\gamma = \inf B$ we have $\inf B \leq \inf A$ (***)

Hence: $\inf B \underset{(***)}{\leq} \inf A \underset{(*)}{\leq} \sup A \underset{(**)}{\leq} \sup B$

(b) $A = [1,3] \cap \mathbb{Q}$ $\quad \inf A = 1, \sup A = 3$

$B = [0,4] \cap (\mathbb{R} \setminus \mathbb{Q})$ $\quad \inf B = 0, \sup B = 4$

$A \cap B = \emptyset$ & $\inf B < \inf A < \sup A < \sup B$

2. (a) Let us fix m $\in \mathbb{Z}$ – the left endpoint of intervals & let $S_m = \{(m,n) ; n > m\}$ $n \in \mathbb{Z}$

Then $$S_m = \bigcup_{k=1}^{\infty} \{(m, m+k)\} \sim \mathbb{N}$$

S_m is equivalent to $\mathbb{N}$ since there is the 1-1 function $\{(m, m+k)\} \longrightarrow k \in \mathbb{N}$

Hence S_m is countable

$$S = \bigcup_{m \in \mathbb{Z}} S_m$$

We know that the countable union of countable sets is countable and since $\mathbb{Z}$ is countable S is countable

(b) The closed interval $[0,1]$ is uncountable.

$A = \{\frac{1}{n} : n \in \mathbb{N}\} \sim \mathbb{N}$ since there is a 1-1 correspondence between $\frac{1}{n} \longrightarrow n$

Suppose $B = [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ is countable.

$[0,1] = A \cup B$ A & B are countable so their sum is also countable and $[0,1]$ is not – contradiction

3 (a) $a_n = \frac{1}{n}\cos n$ $\quad$ $(b_n)_{n\in\mathbb{N}}$ - bounded.

$n = 1, 2 \ldots$

$0 \le |a_n| = |\frac{1}{n}\cos n| \le \frac{1}{n}$

$\frac{1}{n} \to 0$

By the Squeeze Thm. $|a_n| \to 0$

Then $a_n \to 0$ as well : $-|a_n| \le a_n \le |a_n|$, $\quad -|a_n| \to 0$, $|a_n| \to 0$

$(b_n)_{n\in\mathbb{N}}$ is bounded i.e. $\exists\, m, M \in \mathbb{R}$ s.t.

$m \le b_n \le M \quad \forall n \in \mathbb{N}$

Let $B = \max\{|m|, |M|\}$ then $|b_n| \le B$

Take $c_n = a_n \cdot b_n \quad n \in \mathbb{N}$

$$0 \le |c_n| = |a_n| \cdot |b_n| \le \frac{1}{n} \cdot B$$

$\frac{1}{n} \cdot B \to 0 \cdot B = 0$

So $|c_n| \to 0$

By the same argument as with a_n

$c_n \to 0$

$-|c_n| \le c_n \le |c_n|$ $\quad$ so ~~c_n~~

$-|c_n| \to 0$, $|c_n| \to 0$

Sample Final 3 a

$$-\frac{1}{n} \le a_n = \frac{1}{n}\cos n \le \frac{1}{n}$$

$$m \le b_n \le M \quad \text{for any } n \in \mathbb{N}$$

But it is <u>not</u> necessarily true

that $$-\frac{1}{n}\cdot m \le c_n = a_n \cdot b_n \le \frac{1}{n}\cdot M$$

Example:

Let $(b_n)_{n\in\mathbb{N}} = (-1, -100, -1, -100, -1, -100, \ldots)$

Then $m = -100$, $M = -1$

and

$$\underbrace{-\frac{1}{n}(-100)}_{= \frac{100}{n}} \not\le a_n b_n \not\le \frac{1}{n}(-1) = -\frac{1}{n}$$

3 (b) $(a_n)_{n\in\mathbb{N}} = (0,1,0,1, \text{——})$

$(b_n)_{n\in\mathbb{N}} = (1,0,1,0, \text{——})$

Both sequences are divergent since they ~~have~~ contain convergent subsequences with different limits:

$$(a_{2k})_{k=1}^{\infty} = (1,1, \text{——}) \to 1$$

$$(a_{2k+1})_{k=1}^{\infty} = (0,0 \text{——}) \to 0$$

$$(b_{2k})_{k=1}^{\infty} = (0,0 \text{——}) \to 0 \ \& \ (b_{2k+1})_{k=1}^{\infty} = (1, - 1) \to 1$$

$$(a_n+b_n)_{n=\mathbb{N}}^{\infty} = (1,1,1 \text{——}) \xrightarrow[n\to\infty]{} 1$$

(c) $$n\cdot\frac{n}{n^2+n} \le b_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \text{—} + \frac{n}{n^2+n} \le n\cdot\frac{n}{n^2+1}$$

$$n\cdot\frac{n}{n^2+n} = \frac{n^2}{n^2+n} = \frac{1}{1+\frac{1}{n}} \to 1$$

$$n\cdot\frac{n}{n^2+1} = \frac{n^2}{n^2+1} = \frac{1}{1+\frac{1}{n^2}} \to 1$$

So $b_n \to 1$

or $\lim_{n\to\infty} b_n = 1$

4 (a) - - -

(b) $\forall n \quad m \le a_n \le M$

Let $(a_{n_k})_{k=1}^{\infty}$ be any convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ & let $l = \lim_{k\to\infty} a_{n_k}$

Then $\forall k \in \mathbb{N}$

$$\begin{array}{ccccc} m & \le & a_{n_k} & \le & M \\ \downarrow & & \downarrow & & \downarrow \\ m & \le & l & \le & M \end{array}$$

Hence $l \in [m, M]$ & $E \subset [m, M]$

(c) Let $k=1$ since $(a_n)_{n\in\mathbb{N}}$ is not bounded $\exists a_{n_1}$ s.t. $a_{n_1} \le -1$

For $k=2$ $\exists a_{n_2}$ s.t. $a_{n_2} \le \min\{a_{n_1}, -2\}$

then $a_{n_2} \le a_{n_1}$ & $a_{n_2} \le -2$

For $k=3$ $\exists a_{n_3}$ s.t. $a_{n_3} \le \min\{a_{n_2}, -3\}$

then $a_{n_3} \le a_{n_2}$ & $a_{n_3} \le -3$

similarily for any $k \in \mathbb{N}$ $\exists a_{n_k}$ s.t.

4 c

$$a_{n_k} \leq \min\{a_{n_{k-1}}, -k\}$$

then $a_{n_k} \leq a_{n_{k-1}}$ & $a_{n_k} \leq -k$

This subsequence is decreasing.

We want to prove that

$\lim_{k\to\infty} a_{n_k} = -\infty$ i.e. $\forall \beta (<0) \exists n_0 \in \mathbb{N}$

$\forall k \geq n_0 \quad a_{n_k} < \beta$

Let β be fixed. Take n_0 s.t.

$$-n_0 < \beta$$

Then $\forall k \geq n_0 \quad a_{n_k} < -k \leq -n_0 < \beta$

4(d) $a_n = (-1)^n + \frac{n^2-1}{n^2+1}$

For n - odd

$$a_n = -1 + \frac{n^2-1}{n^2+1} = \frac{-n^2-1+n^2-1}{n^2+1} = \frac{-2}{n^2+1}$$

For n - even

$$a_n = 1 + \frac{n^2-1}{n^2+1} = \frac{n^2+1+n^2-1}{n^2+1} = \frac{2n^2}{n^2+1}$$

So $a_{2k+1} = \frac{-2}{(2k+1)^2+1} \longrightarrow 0$

& $a_{2k} = \frac{2(2k)^2}{(2k)^2+1} = 2\frac{1}{1+\frac{1}{(2k)^2}} \longrightarrow 2$

There are 2 subsequential limits:

$E = \{0, 2\}$ so

$\liminf a_n = \inf E = 0$

$\limsup a_n = \sup E = 2$

5 a $\lim_{x \to a} f(x) = L$ i.e.

$\forall \varepsilon_1 > 0 \quad \exists \delta_1 > 0 \quad \forall x \in D_f$

if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \varepsilon_1$

We want to prove that

$\lim_{x \to a} 4f(x) = 4L$ i.e.

$\forall \varepsilon_2 > 0 \quad \exists \delta_2 > 0 \quad \forall x \in D_f$

if $0 < |x-a| < \delta_2$ then $|4f(x) - 4L| < \varepsilon_2$

Let $\varepsilon_2 > 0$

$|4f(x) - 4L| = 4|f(x) - L| < 4 \cdot \varepsilon_1$

For $\varepsilon_1 = \frac{\varepsilon_2}{4}$ we have: $\exists \delta_1 > 0$

$\forall x \in D_f \quad 0 < |x-a| < \delta_1 \quad |f(x) - L| < \varepsilon_1$

Taking $\delta_2 = \delta_1$ we have

$$4|f(x) - L| < 4 \cdot \varepsilon_1 = 4 \frac{\varepsilon_2}{4} = \varepsilon_2$$

for all $x \in D_f$ s.t. $0 < |x-a| < \delta_2$

5 (b)

$$\lim_{x\to c} f(x) = 2 \iff \forall \varepsilon>0 \quad \exists \delta>0$$
$$\forall x\in D_f \quad 0<|x-c|<\delta \quad |f(x)-2|<\varepsilon$$

$$|f(x)-2|<\varepsilon \iff -\varepsilon<f(x)-2<\varepsilon$$
$$2-\varepsilon<f(x)<2+\varepsilon$$

Let $\varepsilon=1$ $\quad \exists \delta>0 \quad \forall x\in D_f \quad 0<|x-c|<\delta$

$$2-1<f(x)<2+1 \iff 1<f(x)<3$$

So $\forall x\in D_f\setminus\{c\}$ s.t. $|x-c|<\delta$ $\quad f(x)>1$

$$|x-c|<\delta \iff c-\delta<x<c+\delta$$

Taking $U=(c-\delta, c+\delta)\cap D_f$ we have
$\forall x\in U\setminus\{c\} \quad f(x)>1$

6 (a) $f: D\to \mathbb{R}$ is continuous at $a\in D$

$\iff \forall \varepsilon>0 \quad \exists\delta>0 \quad \forall x\in D$
if $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$

$\iff \forall (x_n)_{n\in\mathbb{N}} \in D$ if $x_n\to a$
then $f(x_n)\to f(a)$

6 (b)

$$f(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Let f be continuous at x_0 i.e.

$\forall (x_n)_{n \in \mathbb{N}}$ if $x_n \to x_0$ then $f(x_n) \to f(x_0)$

Let $(x_n)_{n \in \mathbb{N}} \in \mathbb{Q}$ s.t. $x_n \to x_0$

& $(y_n)_{n \in \mathbb{N}} \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $y_n \to x_0$

Then since f is continuous at x_0 we have:

$$\begin{array}{ccc} f(x_n) & \longrightarrow & f(x_0) \\ \parallel & & \\ x_n^3 & \longrightarrow & x_0^3 \end{array}$$

and

$$\begin{array}{ccc} f(y_n) & \longrightarrow & f(x_0) \\ \parallel & & \\ y_n & \longrightarrow & x_0 \end{array}$$

f continuous at $x_0 \iff x_0^3 = x_0$

$x_0^3 - x_0 = 0 \iff x_0(x_0^2 - 1) = 0 \iff$

$x_0 = 0$ or $x_0 = \pm 1$

6 (c) $f : [0,1] \to [0,1]$ continuous

$\exists x_0 \in [0,1]$ s.t. $f(x_0) = x_0^2$?

Let $F(x) = f(x) - x^2$

$F(0) = f(0) - 0 = f(0) \geq 0$

$F(1) = f(1) - 1 \leq 0$

If $F(0) = 0$ then $x_0 = 0$ is a solution

If $F(1) = 0$ then $x_0 = 1$ is a solution

Suppose $F(0) \neq 0$ & $F(1) \neq 0$

Then $F(0) > 0$ & $F(1) < 0$

From Intermediate Value Theorem

[$f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$ & $f(a) \neq f(b)$. Then $\forall k$ between $f(a)$ and $f(b)$ $\exists c \in [a,b]$ s.t. $f(c) = k$]

$\exists c \in [0,1]$ s.t. $F(c) = 0$ i.e $f(c) = c^2$

Taking $x_0 = c$ we have the solution

7. f is uniformly continuous on $D \Leftrightarrow$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x_1, x_2 \in D$ if $|x_1 - x_2| < \delta$

then $|f(x_1) - f(x_2)| < \varepsilon$

(a) $f(x) = x^2$

Let $\varepsilon > 0$ & $x_1, x_2 \in (0, a]$

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)| \underset{x_1 > 0}{\overset{x_2 > 0}{=}}$$

$$= |x_1 - x_2|(x_1 + x_2) \underset{\substack{x_1 \leq a \\ x_2 \leq a}}{\leq} 2a \cdot |x_1 - x_2|$$

Take $\delta = \frac{\varepsilon}{2a}$ For $x_1, x_2 \in (0, a]$ if

$|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| \leq 2a\,|x_1 - x_2| <$

$$< 2a \cdot \delta = 2a \cdot \frac{\varepsilon}{2a} = \varepsilon$$

so $f(x) = x^2$ is uniformly cont. on $(0, a]$

(b) $f(x) = 1/x^2$ let $x_1, x_2 \in (0, a]$

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = \frac{|x_2^2 - x_1^2|}{x_1^2 \cdot x_2^2}$$

Let $\delta > 0$, $x_1 = \delta$ & $x_2 = \frac{\delta}{2}$ then $|x_1 - x_2| = \frac{\delta}{2} < \delta$

and $|f(x_1) - f(x_2)| = \dfrac{3/4\,\delta^2}{\frac{1}{4}\delta^4} = \dfrac{3}{\delta^2}$

For $\varepsilon < \frac{3}{\delta^2}$ we have $|x_1 - x_2| < \delta$ &

$|f(x_1) - f(x_2)| = \frac{3}{\delta^2} > \varepsilon$ f is not unif. cont.

8 a $\quad f(x) = |x^3| = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $x > 0$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh) = 3x^2$$

For $x < 0$

$$f'(x) = \lim_{h \to 0} \frac{-(x+h)^3 - (-x^3)}{h} = \lim_{h \to 0} \frac{\cancel{-x^3} - 3x^2h - 3xh^2 + \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} (-3x^2 - 3xh) = -3x^2$$

For $x = 0$

$$\lim_{h \to 0^+} \frac{h^3 - 0}{h} = 0 \qquad \text{So } f'(0) = 0$$

$$\lim_{h \to 0^-} \frac{-h^3}{h} = 0 \qquad f'(x) = \begin{cases} 3x^2 & x > 0 \\ -3x^2 & x < 0 \\ 0 & x = 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & x > 0 \\ -6x & x < 0 \\ 0 & x = 0 \end{cases} \qquad \lim_{h \to 0^+} \frac{3h^2}{h} = 0 = \lim_{h \to 0^-} \frac{-3h^2}{h}$$

8 (b) Prove $\ln(1+x) < x$ for $x > 0$

Fix $x > 0$ & $f(x) = \ln(1+x)$

Consider f on the interval $[0, x]$

f is continuous on $[0, x]$ & differentiable on $(0, x)$

$$f'(y) = \frac{1}{1+y}$$

From the M.V. Thm. $\exists\, c \in (0, x)$

s.t. $f(x) - f(0) = f'(c)(x - 0)$

i.e.

$$\ln(1+x) - 0 = \frac{1}{1+c} \cdot x \underset{c>0}{<} x$$

Hence

$$\ln(1+x) < x$$

Bonus

$f: D \to \mathbb{R}$, $a \in D$. f is continuous at $a \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D$ if $|x-a| < \delta$ then

$|f(x) - f(a)| < \varepsilon \Leftrightarrow \forall (x_n)_{n \in \mathbb{N}} \in D$ s.t.

$x_n \to a$, $f(x_n) \to f(a)$

"$\Rightarrow$" Let $\varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D$

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Let x_n be any seq. s.t. $x_n \to a$ i.e.

$\exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ |x_n - a| < \delta$

So for $n \geq n_0$ $|f(x_n) - f(a)| < \varepsilon$ i.e.

$f(x_n) \to f(a)$

"$\Leftarrow$" Suppose f is not continuous at a i.e.

$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in D \ |x-a| < \delta \ \& \ |f(x) - f(a)| \geq \varepsilon$

For any $n \in \mathbb{N}$ take $\delta_n = \frac{1}{n}$

$\forall n \in \mathbb{N} \ \exists x_n \in D$ s.t. $|x_n - a| < \frac{1}{n}$

but $|f(x_n) - f(a)| \geq \varepsilon$

We have $(x_n)_{n \in \mathbb{N}} \in D$ $x_n \to a$

but $f(x_n) \not\to f(a)$

contradiction