MATH 364, December 2012 Final, SOLUTIONS:

Problem 1: Let S be a non-empty subset of $\mathbb{R}$ which is bounded above and denote $\alpha = \sup S$.

(a) Prove that for any $\varepsilon > 0$, there exists $s \in S$ such that $\alpha - \varepsilon < s \leq \alpha$.

Solution: By contradiction. Let us assume that this is not true, i.e., there exists an $\varepsilon > 0$ such that for all $s \in S$ we have $s \leq \alpha - \varepsilon$ or $s > \alpha$. Since $s > \alpha$ is not possible (α is the supremum of S) we must have

$$\text{for all } s \in S : \ s \leq \alpha - \varepsilon.$$

Then, $\sup S \leq \alpha - \varepsilon$, which contradicts the definition of α.

(b) Prove that there exists a sequence $\{s_n\}_{n=1}^{\infty} \subset S$ such that $s_n \to \alpha$.

Solution: Using part (a) we can find for any $n = 1, 2, 3, \ldots$ an element $s_n \in S$ such that

$$\alpha - 1/n < s_n \leq \alpha \ .$$

By "sandwich theorem" $s_n \to \alpha$.

Problem 2:

(a) Prove that $f(x) = 2x + 5$ is uniformly continuous on $\mathbb{R}$.

Solution: We have

$$|f(x) - f(y)| = |2x + 5 - 2y - 5| \leq 2|x - y| \ ,$$

for all $x, y \in \mathbb{R}$. Thus, f satisfies Lipschitz condition and is uniformly continuous on $\mathbb{R}$.

(b) Suppose that a function f is uniformly continuous on an interval D. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in D. Prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

Solution: We need to show

$$\forall_{\varepsilon>0} \ \exists_{\ N\geq 1} \ \forall_{\ n,m\geq N} \ |f(x_n) - f(x_m)| < \varepsilon \ .$$

Let us fix an $\varepsilon > 0$. Since f is uniformly continuous for this ε we can find a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \ ,$$

for all $x, y \in D$.

Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy we can find an $N \geq 1$ such that for all $n, m \geq N$ we have

$$|x_n - x_m| < \delta \ .$$

Then, if $n, m \geq N$, we have

$$|f(x_n) - f(x_m)| < \varepsilon \ .$$

This proves that $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

(c) Prove that $f(x) = 1/x^4$ is not uniformly continuous on $(0, 1]$.

Solution: The sequence $\{1/n\}_{n\geq 1} \subset (0, 1]$ is Cauchy (it converges). The sequence $f(1/n) = n^4$ is not Cauchy, as it diverges to infinity. In view of (b), f cannot be uniformly continuous.

(d) Prove that $f(x) = 1/x^4$ is uniformly continuous on $[1, \infty)$.

Solution: By Mean Value Theorem:

$$|f(x) - f(y)| = |f'(c)||(x - y)| \ ,$$

with c between x and y. We have

$$\left|\frac{1}{x^4} - \frac{1}{y^4}\right| = \left|\frac{-4}{c^5}\right| |x-y| \leq 4 \cdot |x-y| \ ,$$

so f satisfies Lipschitz condition in $[1, \infty)$ and thus it is uniformly continuous.

Problem 3:

(a) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in the interval $[a, \infty)$. Prove that if x_n converges to x, then $x \geq a$.

Solution: By contradiction: Let us assume that $x < a$. Take $\varepsilon = (a-x)/3 > 0$. By the definition of the limit, for this ε we can find an $N \geq 1$ such that for any $n \geq N$ we have $x - \varepsilon < x_n < x + \varepsilon$. In particular, $x_n < x + (a-x)/3 < a$ which contradicts the assumption $x_n \geq a$.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at c, and that $f(c) > 0$. Prove that there is a neighbourhood U of c such that $f(x) > 0$ for all $x \in U$.

Solution: Let $\varepsilon = f(c)/2 > 0$. By the definition of the continuity at c, there exists a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$ we have

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \ .$$

In particular,

$$f(x) > f(c) - f(c)/2 = f(c)/2 > 0 \ .$$

(c) Use the Intermediate Value Theorem to show that the function $f(x) = 3x^4 + x - 7$ has a root in the interval $[1, 2]$.

Solution: f is a polynomial so it is continuous on $\mathbb{R}$. We have $f(1) = 3+1-7 < 0$ and $f(2) = 3 \cdot 16 + 2 - 7 > 0$. By Intermediate Value theorem f has a root between 1 and 2.

Additional remark: Since $f'(x) = 12x^3 + 1 > 0$, for $x \in [1, 2]$, f is strictly increasing in $[1, 2]$ so it has exactly one root in this interval.

Problem 4:

(a) Suppose S is an infinite subset of $\mathbb{R}$. Prove that S contains a sequence of distinct points.

Solution: First method: Since S is infinite its cardinality is at least $\aleph_0$. This means that there exists an injection $g : \mathbb{N} \to S$. The sequence is $\{g(n)\}_{n=1}^{\infty}$. Its elements are distinct since g is an injection.

Second method: We will construct the sequence by induction. Let s_1 be any element of S. Let us assume that we already constructed elements $s_1, s_2, \ldots, s_n$. Since S is infinite the set $A_n = S \setminus \{s_1, s_2, \ldots, s_n\}$ is not empty. Let s_{n+1} be any element of A_n. It is different from any s_i with $i \leq n$. This constructs the required sequence.

(b) Suppose that S is a subset of $\mathbb{R}$ which is unbounded above. Prove that S contains a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to \infty$.

Solution: Since S is unbounded above for any $n \geq 1$ it contains an element $s_n > n$. The sequence $\{s_n\}_{n=1}^{\infty}$ (we do not care if the elements repeat or not) satisfies

$$n < s_n \quad , \quad n = 1, 2, \ldots$$

so $s_n \to \infty$.

(c) Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded above. Prove that it contains a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \infty$.

Solution: Now, we have to be more careful as the indices cannot repeat. $\{x_n\}_{n=1}^{\infty}$ is unbounded above so it contains an element $x_{n_1} > 1$. Assume that the elements $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ satisfying $n_1 < n_2 < \cdots < n_k$ and $x_{n_i} > i$, $i = 1, 2, \ldots, k$ has been found. The sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded above so it contains an elements $x_j > k+1$ with arbitrary large indices j. Let $x_{n_{k+1}}$ satisfies $n_{k+1} > n_k$ and $x_{n_{k+1}} > k+1$. This constructs a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ satisfying

$$k < x_{n_k} \quad , \quad k = 1, 2, \ldots$$

which diverges to infinity.

Problem 5:

(a) Suppose $\{x_n\}_{n=1}^{\infty}$ is a nonconvergent sequence. Prove that for any $x \in \mathbb{R}$ there exists $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathbb{N}$.

Solution: Fix an $x \in \mathbb{R}$. Since $x_n \not\to x$, we have

$$\exists_{\ \varepsilon > 0}\ \forall_{\ N \geq 1}\ \exists_{\ n \geq N}\ |x - x_n| \geq \varepsilon\ .$$

Using this we construct subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ inductively:

For $N = 1$ there exists x_n with $n \geq 1$ such that $|x - x_n| \geq \varepsilon$. We call it x_{n_1}.

For $N = n_1 + 1$ there exists x_n with $n \geq n_1 + 1$ such that $|x - x_n| \geq \varepsilon$. We call it x_{n_2}.

If $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ are already chosen, then for $N = n_k + 1$ there exists x_n with $n \geq n_k + 1$ such that $|x - x_n| \geq \varepsilon$. We call it $x_{n_{k+1}}$,etc.

This way we construct the required subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

(b) Also show that there are at least two subsequences of $\{x_n\}_{n=1}^{\infty}$ with distinct limits.

Solution: Since $\{x_n\}_{n=1}^{\infty}$ is a nonconvergent sequence, we have $\alpha = \liminf x_n < \limsup x_n = \beta$. On the other hand we proved that $\{x_n\}_{n=1}^{\infty}$ contains subsequences convergent to α and to β. These are two subsequences with distinct limits.

Problem 6: Use induction to prove

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)\ .$$

Solution: For $n = 1$ we have equality $1^2 = \frac{1}{6}(1)(1+1)(2+1)$. Let us assume the equality for n. We have

$$\frac{1}{6} n(n+1)(2n+1) + (n+1)^2 = \frac{(n+1)}{6}[n(2n+1) + 6(n+1)] = \frac{(n+1)}{6}[(n+2)(2(n+1)+1)]\ .$$

Thus, using inductive assumption we obtain

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2(n+1)+1)\ ,$$

which is the formula for $n+1$. By the principle of mathematical induction the formula holds for all $n \in \mathbb{N}$.

Problem 7: Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^4 \sin\left(\frac{5}{x^2+x^3}\right) & \text{if} \quad x \neq 0 ; \\ 0 & \text{if} \quad x = 0 . \end{cases}$$

Prove that f is differentiable at $x = 0$ and find $f'(0)$.

Solution: By the definition of the derivative

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h^3 \sin\left(\frac{5}{h^2 + h^3}\right) = 0 ,$$

since sin is bounded between -1 and 1.

Problem 8: Find the Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x}$ about the point $x = 1$ and state the formula for the remainder, $R_3(x)$.

Solution: Taylor formula for $n = 3$ at $x = 1$:

$$f(x) = f(1) + f'(1)(x-1) + \frac{1}{2} f''(1)(x-1)^2 + \frac{1}{6} f'''(1)(x-1)^3 + \frac{1}{24} f^{(IV)}(c)(x-1)^4 ,$$

where c is a point between x and 1.

We have:
$f'(x) = \frac{1}{2} x^{-1/2}$,
$f''(x) = \frac{1}{2}\frac{-1}{2} x^{-3/2} = \frac{-1}{4} x^{-3/2}$,
$f'''(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2} x^{-5/2} = \frac{3}{8} x^{-5/2}$, and
$f^{(IV)}(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2} x^{-7/2} = \frac{-15}{16} x^{-7/2}$. Thus,

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^3 - \frac{15}{24 \cdot 16} c^{-7/2}(x-1)^4 ,$$

with c between x and 1.

Problem 9: Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. Prove that f is monotone increasing on $\mathbb{R}$ if and only if $f'(x) \geq 0$ for all $x \in \mathbb{R}$.

Solution: "$\Rightarrow$": If f is increasing, then for any $x \in \mathbb{R}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \geq 0 ,$$

since $f(x+h) - f(x)$ and h are of the same sign.

"$\Leftarrow$": If $f'(x) \geq 0$ for all $x \in \mathbb{R}$, then for $a < b$ we have by Mean Value Theorem:

$$f(b) - f(a) = f'(c)(b-a) \geq 0 ,$$

and f is increasing.

Problem 10: Prove

$$\ln \frac{a}{b} \leq \frac{a-b}{b} \quad , \quad \text{for} \quad 0 < b < a .$$

Solution: Using Mean Value Theorem with c such that $b < c < a$:

$$\ln\frac{a}{b} = \ln a - \ln b = \frac{1}{c}(a-b) \leq \frac{1}{b}(a-b) = \frac{a-b}{b}\ .$$